PROCEEDINGS
OF THE
NATIONAL ACADEMY OF SCIENCES
INDIA
1952

PART I] SECTION A [VOL. 21

ISOLATION OF ABIETIC ACID FROM ROSIN

By

S. P. MITRA

SHEILA DHAR INSTITUTE OF SOIL SCIENCE, UNIVERSITY OF ALLAHABAD

Read on 26th April, 1951

INTRODUCTION

The work on the chemistry of rosin dates back to 1826 when Baupe[1] first showed that it contained crystallizable acids. The advance in the chemistry of these exudates (oleoresins) has been rapid. This is probably due to the scientific interest in the difficulty of correctly ascertaining their structures as well as to the commercial interest in developing them as cheap raw materials. A large amount of work has since been published on the rosin acids and on their relation to the original composition of rosin. It is only in recent years that some degree of order has been evolved in this difficult field of research. As a result, it has been established that the rosin acids belong to two different groups: (a) the abietic acid group which when dehydrogenated with sulphur yield retene and (b) the pimaric acid group which on dehydrogenation yield a hydrocarbon $C_{16}H_{14}$ probably a dimethyl phenanthrene. The empirical formula of these acids were first determined as $C_{20}H_{30}O_2$ by Trammsdorf[2] who analyzed the copper salt, which formula has now been accepted.

Abietic acid is not an original constituent of tree secretions and it apparently is not present as such to an appreciable extent in rosin,

It arises as a transformation product of the natural primary acids in the course of preparation and further treatment of rosin possibly through a succession of intermediate forms. Rosin appears to be a mixture of acids in various stages of transformation. Changes occur during storage and under the influence of mild heat. It is not an easy matter to determine if a substance isolated even under the mildest condition is a primary constituent. Hasslestrom and Bogert[3] suggest that the fresh oleoresin contain 3 acids; d-pimaric acid (which has no structural relation with l-pimaric acid), l-pimaric acid and l-sapietic acid. It thus appears that pine oleoresin is a mixture of labile sapinic acids which are converted on distillation to a mixture of more stable acids some of which are isomeric including abietic acid.

Rau and Simonsen[4] have established the nature of the chief acid present in Indian rosin derived from *Pinus longifolia* and have shown it to be abietic acid of the empirical formula $C_{19}H_{29}COOH$.

EXPERIMENTAL

In an attempt to find out a more economical method for the manufacture of abietic acid from Indian rosin than is possible by any of the existing methods namely: acetic acid method of Steele[5], high vacuum distillation method of Ruzicka and Meyer[6], crystalline acid salt method of Dupont, Desalbres and Bernette[7] and Palkin and Harris[8], hydrochloric acid gas method of Keseler, Lowry and Faragher[9], diamylamine salt method of Harris and Sanderson[10], alcoholic hydrochloric acid method of Siddiqui, Bose, Das Gupta and Chakravorty[11] and steam and hydrochloric acid gas method of Shkatelov[12], Indian rosin was subjected to treatment with high temperature steam. There are isolated references to this method[13] for the preparation of abietic acid but they have not been carried to completion anywhere. Moreover, no work has been done with Indian rosin on the above lines.

An apparatus was fabricated in which rosin could be treated continuously with high temperature steam at any desired temperature.

The apparatus consists of a steam generator connected with a copper spiral tube dipping in a cylindrical iron vessel which is heated electrically. The cylindrical vessel contains an eutectic mixture of

Sodium Nitrite	40%
Sodium Nitrate	7%
and Potassium Nitrate	53%.

This serves as an inorganic salt bath which can be raised to a temperature of 600°C. On the lid of the cylindrical vessel is a thermometer pocket in which a thermometer or pyrometer is put to record the temperature of the salt bath. The steam from the generator during its passage through the spiral tube dipping in the salt bath takes up the temperature of the bath. The temperature of the exit steam was measured by placing a thermometer in a pocket at the outlet.

Best quality Indian Rosin (water white grade) from *Pinus longifolia* from Indian Rosin and Turpentine factory, Cultterbuckgunj (Bareilly) ($[\alpha]_D^{25}$-7·48 in ethyl alcohol) was subjected to this treatment for definite periods of time. Care was taken to use freshly powdered rosin every time. The rosin was maintained in a molten state in a pyrex flask at the same temperature at which superheated steam is led into it. It was observed that upto a temperature of 200°C only a little oily liquid (probably terpenes) distill over. This constitutes about 2·3 per cent of the total rosin treated. The steam treated rosin thus obtained were pale in colour and crystals of abietic acid separate out easily from solutions in 90% ethyl alcohol.

Anal:	Calculated for $C_{20}H_{30}O_2$.	C-79·47	H-9·93
	Found	C-79·21	H-10·73

A number of derivatives were prepared and their melting points were determined. The results compared to the melting points of the same derivatives of abietic acid (prepared by different methods) by other workers are given in Table I.

TABLE I

	Derivatives	Ruzicka & Meyer	Aschan & Virtanen	Rau & Simonsen	Mitra
1.	Dibromo acid	—	M.P. 107-110°C	M.P. 108°C	M.P. 110-11°C
2.	Dihydrobromide	M.P. 178°C	M.P. 188-92°C	M.P. 189°C	M.P. 190-92°C
3.	Monohydroiodide	—	M.P. 191-93°C	M.P. 191°C	M.P. 189-91°C
4.	Monohydrochloride	—	—	M.P. 197°C	M.P. 198-99°C
5.	Nitrosochloride	—	M.P. 144-45°C	M.P. 144°C	M.P. 143-45°C
6.	Monohydroxy acid	—	—	M.P. 230°C	M.P. 231-33°C

The results obtained when rosin is treated at different temperatures for varying lengths of time are shown in table II.

It was further observed that on progressively raising the temperature of the steam above 200°C, a yellow solid product began to distill over. The quantity of the distillate increased with temperature until a temperature of 300°C was reached when there was tendency for the major portion of the rosin to distill over. On exhaustive distillation about 60 per cent yellow transparent solid material was obtained. During the first stages of distillation, the distillate was bright yellow in colour with a soft consistency which changed to paler colour and harder consistency with the progress of distillation. Abietic acid could be obtained from the yellow product by crystallization from dilute alcoholic solutions. A second crop of crystals were obtained by concentrating the filtrate.

The results obtained are given in table III.

It can be seen from the table that the percentage of acid obtained from the residue treated at higher temperature is always less than that obtained from the residue treated at lower temperature. On the other hand the amount of abietic acid obtained from the distillate goes on increasing with increase of temperature. Beyond 300° C

TABLE II

Quantity of rosin taken in each experiment=100 grams

Temperature	2 Hours			4 Hours			6 Hours			8 Hours		
°C	Yield %	M. P. °C	$[\alpha]^{25}_{D}$	Yield %	M. P. °C	$[\alpha]^{25}_{D}$	Yield %	M. P. °C	$[\alpha]^{25}_{D}$	Yield %	M. P. °C	$[\alpha]^{25}_{D}$
150°	10·12	150-52	—50·24	14·26	151-52	—51·76	18·74	154-55	—59·28	19·69	150-53	—53·46
160°	12·36	155-56	—51·76	16·72	152-54	—57·28	19·30	158-59	—52·46	19·59	152-55	—50·08
170°	13·14	152-55	—55·11	18·73	153-54	—52·26	20·46	153-155	—53·48	21·67	156-57	—50·97
180°	16·61	150-52	—51·28	23·49	155-58	—56·68	29·60	156-57	—52·44	31·78	154-56	—53·98
190°	20·24	152-53	—51·28	28·85	156-59	—60·28	34·26	154-56	—52·44	36·17	158-59	—57·38
200°	25·71	156-57	—55·27	38·22	152-54	—50·71	46·89	155-58	—55·89	47·23	152-54	—52·67

TABLE III

Quantity of rosin taken each time=100 grams
Time=5 Hours

Temperature °C	Distillate in grams	Residue in grams	Acid from distillate in grams	Acid from residue in grams	Total abietic acid %	M. P. °C	$[\alpha]_D^{25}$
210	14·22	84·20	3·60	38·64	42·24	150-52	—52·24
220	19·32	79·42	6·46	38·78	45·24	156-57	—50·68
230	24·74	73·23	12·49	37·68	50·17	154-56	—59·33
240	26·82	71·75	20·71	28·57	49·28	155-59	—57·11
250	29·71	69·29	26·51	27·16	53·67	149-53	—50·20
260	32·47	62·46	29·22	23·74	52·96	158-59	—58·76
270	43·69	55·17	38·64	19·58	58·22	151-54	—55·11
280	49·28	47·62	46·34	5·52	51·76	157-59	—59·31
290	54·76	41·22	52·04	1·46	53·50	151-56	—52·44
300	66·32	30·47	54·72	1·22	55·94	152-53	—50·78

decomposition of the rosin sets in and copious fumes are given off. Although at 300°C the maximum amount of abietic acid is obtained from the distillate but the total amount of abietic acid obtained from both distillate and residue is maximum at 270°C when 58·22% of abietic acid is obtained. It is not advisable moreover to go beyond 300°C for then abietic acid is converted into pyroabietic acid.

Discussion

The form in which abietic acid exists in rosin has been a matter of great controversy for a long time. Maly[14] first postulated that the acid exists as anhydride and was supported by Bischoff and Nastvogel[15]. But Henrique[16], and Easterfield and Bagley[17] have not been able to obtain an anhydride by high vacuum distillation of rosin as reported by Bischoff and Nastvogel. Ruzicka and Meyer[18] and Ruzicka and Schinz[19] were unable to verify the work of Knecht and Hibbert[20] who have advanced the anhydride theory that abietic acid was dehydrated by heating at 190°C in carbon dioxide for 8 hours. Steele[21] has also postulated that the acid exists as anhydride for he has not been able to obtain abietic acid from rosin dissolved in 98% acetic acid in cold. But abietic acid crystallized out when the solution was allowed to stand after being refluxed for 2 hours during which time hydration was supposed to occur. Schroger[22] obtained abietic acid by crystallization of vacuum distilled rosin from anhydrous petroleum ether. This refutes the work of Knecht, Steele and others who postulate that rosin is composed essentially of anhydrides but supports the "Isomerization theory" which postulates that abietic acid exists as free acid in rosin which has to undergo an isomerization before it can be crystallized out from solutions in organic solvents. Shaw and Sebrell[23] have also produced strong evidence against the anhydride theory.

Keseler, Lowry and Faragher[24] in their studies on the preparation of abietic acid by the crystalline acid salt method have found that isomerization of rosin is essential for the formation of complex salts as commercial amorphous rosin gives no such precipitate. The isomerization can be followed by the change in optical rotation. The

original acid rotates light to the right while the isomerized product gives a negative rotation. Mineral acids also effect this change in optical rotation. Aqueous or gaseous hydrochloric acid proves to be an effective isomerizing agent. The same result may also be obtained by heating to 250°C.

It was shown that heat or mineral acids cause the isomerization of primary rosin acids (except d-pimaric acid) to secondary acids and these secondary acids are in turn isomerized to abietic acid. Abietic acid, however, must not be considered the final isomerization product, since it can be isomerized to pyro-abietic acid. According to Dupont[25] the isomerization proceeds as follows:—

Primary acids→Secondary acids→Abietic acid→Pyroabietic acid.

It remained for Kohler[26] to present the first classification of rosin acids. It is generally conceded that the isomerization process is caused by a shifting of the position of the double bonds in the rosin acid molecule from a position of higher to one of lower free energy[27]. The ratio of these different rosin acids to one another in any given grade of rosin will vary considerably because of the greater number of factors affecting the isomerization.

In order to ascertain the exact role that superheated steam plays in the above process, experiments were done using inert gases like carbon dioxide and nitrogen in place of superheated steam. It was found that when carbon dioxide at a temperature of 200°C was bubbled through molten rosin maintained at that temperature, at first a colourless liquid (water) collect in the receiver which constitute about 1·2% of the total rosin treated. After some time light yellow oily liquid (terpenes) collect in the receiver constituting about 1·4% of the total rosin treated. The heat treated rosin thus obtained were of a pale colour and abietic acid crystallized from it from alcoholic solutions

M. P. 151-54°C $\left[\alpha\right]_{D}^{25°}$ −53·26

On gradually raising the temperature of the gas to 300°C, a yellow solid product began to distill over. Abietic acid crystallized out from the light yellow solid from alcoholic solutions.

M. P. 150-55°C $\left[\alpha\right]_D^{25°}$ $-59{\cdot}37$

The only difference observed in this case was that the distillation of yellow solid began at about 300°C whereas with superheated steam it began at 200°C. The yield of abietic acid from carbon dioxide treated rosin was less than that obtained from steam treated rosin at the same temperature.

The fact that abietic acid can be obtained by mild heat treatment of rosin in a suitable inert atmosphere seems to lend support to the "free acid theory" rather than "Anhydride theory" in the agelong controversy on the occurrence of abietic acid in rosin. Abietic acid exists in a free state in rosin and the application of heat isomerizes it, which then can be crystallized out from solutions. In this way only can we reconcile the contradictory views of Steele on the one hand and Ruzicka and Meyer on the other. But prolonged heat treatment at very high temperatures carries the isomerization one step further and pyro-abietic acids are produced. In all these methods for the isolation of abietic acid from rosin heating is done at some stage which is responsible for isomerization of abietic acid which could thus be crystallized out from alcoholic solutions.

Summary

Abietic acid has been isolated from Indian rosin by treatment of superheated steam below 200°C. Between 200-300°C a yellow solid product began to distill over which also gave crystals of abietic acid from solutions in alcohol. On raising the temperature beyond 300°C decomposition of the rosin sets in and copious fumes are given off. Abietic acid was also obtained from rosin by treating it with carbon dioxide below 300°C. In this case distillation of yellow solid product began at 300°C. The yield of abietic acid was more

when treated with high temperature steam than when treated with high temperature carbon dioxide.

References

1. Baupe — *Ann. chim phys.* 1826, page 108.
2. Trammsdorf — *Ann. Pharm.* **13**, 169 (1835)
3. Hasslestrom and Bogert — *J. Am. Chem. Soc.* **57**, 2118 (1935)
4. Rau and Simonsen — *Ind. Forest Records* **11**, 1 (1924)
5. Steele — *J. Am. Chem. Soc.* **44**, 1333 (1922)
6. Ruzicka and Meyer — *Helv. Chim. Acta* **5**, 315 (1922)
7. Dupont, Desalbres and Bernette — *Bull. Inst. du. Pin* **22**, 349 (1926)
8. Palkin and Harris — *J. Am. Chem. Soc.* **56**, 1935 (1934)
9. Keseler, Lowry and Faragher — *Ibid* **49**, 2898 (1927)
10. Harris and Sanderson — *Ibid* **70**, 334 (1948)
11. Siddiqui, Bose, Das Gupta and Chakravorty — *J. Sci and Ind. Res* **78**, 149 (1948)
12. Shkatelov — *Lesokhimicheskaja Prom* **3**, 13 (1934)
13. Easterfield and Bagley — *J. Chem. Soc.* **85**, 1238 (1904)
14. Maly — *Ann* **149**, 244 (1869)
15. Bischoff and Nastvogel — *Ber* **23**, 1921 (1890)
16. Henrique — *Chem. Reviews* 1899 page 106
17. Easterfield and Bagley — *loc. cit.*
18. Ruzicka and Meyer — *Helv. Chim. Acta* **5**, 315 (1922)
19. Ruzicka and Schinz — *Ibid* **6**, 833 (1923)
20. Knecht and Hibbert — *J. Soc. Dyers and Colourists* **35**, 149 (1919), **39**, 338 (1923)
21. Steele — *loc. cit.*
22. Schorger — *J. Am Chem. Soc.* **45**, 1239 (1923)
23. Shaw and Sebrell — *Ind. Eng. chem.* **18**, 612 (1926)
24. Keseler, Lowry and Faragher — *loc. cit*
25. Dupont — *Bull. Soc. chim.* **29**, 718 (1921)
26. Kohler — *J. Prakt. Chem.* **85**, 534 (1912)
27. Ruzicka — *Bull Inst. du. Pin* **59**, 112 (1929)

THE 's-t' DOUBLET IN THE MERCURY L X-RAY SPECTRUM*

By

CHINTAMANI MANDE, M.Sc.

DEPARTMENT OF PHYSICS, ALLAHABAD UNIVERSITY, ALLAHABAD.

[Communicated by Dr. G. B. Deodhar, Ph.D. (Lond), F. Inst. P. (Lond), F.N.A.Sc.]

ABSTRACT

Two new lines $\lambda 1308{\cdot}5$ X.U. and $\lambda 1371{\cdot}8$ X.U. have been discovered in the L-emission spectrum of mercury. It has been shown that these are the forbidden lines 's' and 't' corresponding to the transitions $L_{III}M_{III}$ & $L_{III}M_{II}$. It is also shown that these lines form a regular doublet.

INTRODUCTION

The 's' and 't' lines corresponding to the forbidden transitions $L_{III}M_{III}$ and $L_{III}M_{II}$ respectively have been reported in the L-emission spectra of some elements of high atomic number by various workers. Thus the line 's' has been reported for 73 Ta, 74W and 77 Ir by Auger and Dauvillier[1], for 78 Pt by Deodhar and the author[2], for 79Au by Auger and Dauvillier, and Woodall[3], and for 82 Pb and 83 Bi by Idei[4]. Similarly the line 't' has been reported for 73Ta, 74W, 77Ir, 78Pt and 79Au by Auger and Dauvillier, for 78Pt by Deodhar and the author, for 82Pb by Eddy and Turner[5], and Idei and for 83Bi by Eddy and Turner.

The line 's' corresponding to the transition $L_{III}M_{III}$ has not so far been observed by any worker in the mercury L-spectrum. A line $\lambda 1380{\cdot}6$ X.U. has been measured by Eddy and Turner[6] while studying the mercury L-spectrum and has been tentatively assigned by them as the line 't' corresponding to the transition $L_{III}M_{II}$. However, Eddy and Turner themselves have pointed out that this line does not fit in the Moseley graph of the line 't'. It is therefore doubtful whether the line

* This paper was read in a general meeting of the Nat. Acad. of Sci. held on April 26, 1951.

λ1380·6X.U. measured by Eddy and Turner really corresponds to the L_{III} M_{II} transition of mercury. It was therefore considered worthwhile to search for the 's' and 't' lines in the mercury L-spectrum in this laboratory.

EXPERIMENTAL

In studying the X-ray emission spectra of mercury considerable difficulty is met in preparing a mercury surface which would last for several hours. However, it was found that a satisfactory coating is obtained by wetting a copper anti-cathode with mercury nitrate solution and then leaving the surface standing in liquid mercury for about 24 hours. The target is then removed and left in air for several days. This method gives a dry amalgam, which lasts for about 4 to 5 hours satisfactorily. After about 5 hours, however, the anticathode has to be taken out and is replaced by another one. This, of course, means disturbing the vacuum everytime, but never-the-less it is very important. If the anticathode is not replaced by a new one, the general blackening due to the copper target on the film would make it impossible to distinguish faint mercury lines.

The X-ray tube was of the metal type constructed in this laboratory, after a design by Siegbahn for spectroscopic work. The cathode, the anticathode and the body of the X-ray tube were cooled by a current of water running continuously. The tube was excited by a step up transformer connected to a mechanical rectifier. The voltage applied was between 40 to 50 KV and the tube current was maintained at about 5 m. amps.

The spectrometer was a single crystal instrument of medium dispersion; the distance between the slit and the crystal being 25cms, and the distance between the crystal and the film about 20 cms. A calcite crystal of high degree of perfection was used for this investigation.

By giving varying exposures to the extent of 50 hours on a single film, several spectrograms were obtained. However, it was found that the 's' and 't' lines appeared only on giving heavy exposure.

The reference lines used for measurements were Hg $L\alpha_1$, for which Friman's[7] value $\lambda = 1238{\cdot}63$X.U. was taken, and Cu $K\beta_1$ for which Wennerlöf's[8], value $\lambda = 1389{\cdot}35$X.U. was taken.

Results and Discussion

Table I gives the wave-length measurements and ν/R values obtained for these lines by the author.

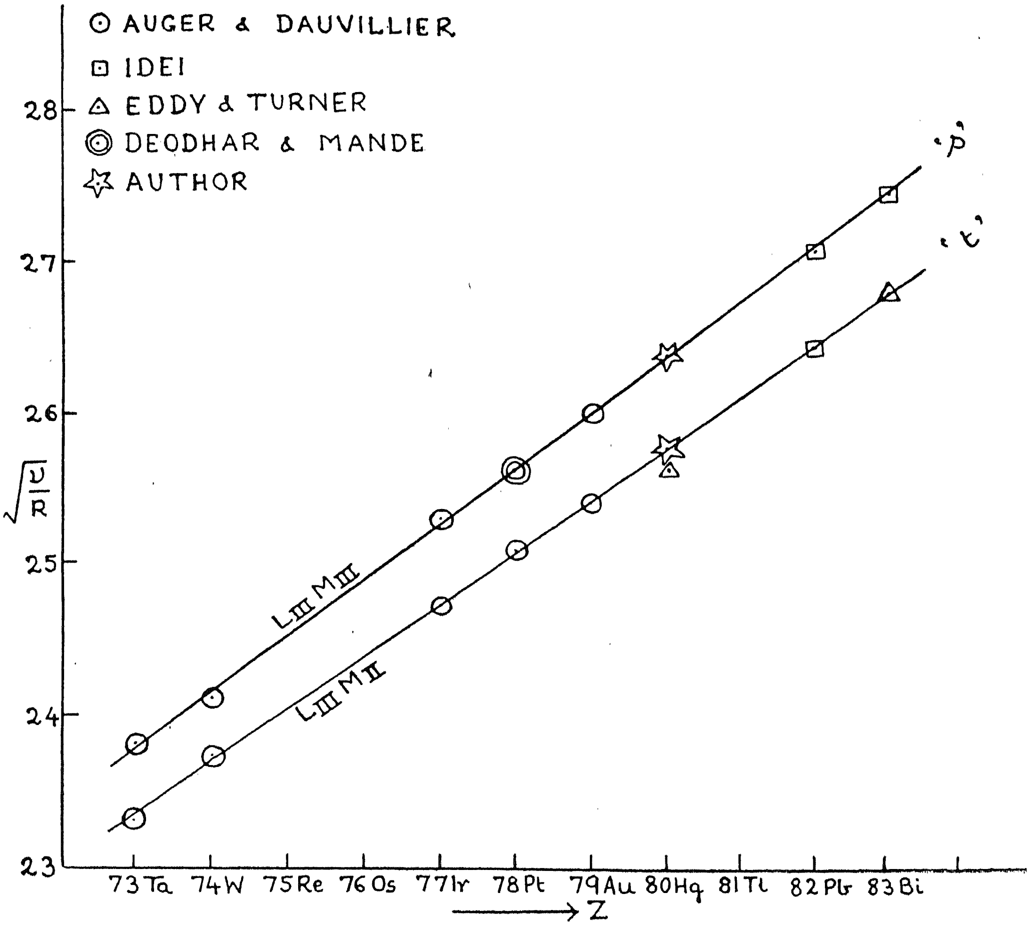


Fig. 1, showing the Moseley graphs of the lines *s* and *t*

TABLE I

Line	Transition	Δl	Δj	λ in X.U.	ν/R
's'	$L_{III}\ M_{III}$	0	0	1308·5	696·4(2)
't'	$L_{III}\ M_{II}$	0	+1	1371·8	664·2(9)

Moseley diagrams of these lines have been shown in fig. 1 for elements between 73 Ta and 83 Bi . It can be easily seen that the ν/R values for these lines obtained by the author fit very well in these curves, so that no doubt is left about the correct assignment of transitions for the two lines measured by the author. The $\sqrt{\nu/R}$ value of the line λ1380·6 measured by Eddy and Turner has also been shown in Fig 1. It may be seen that this line lies away from the graph of the line 't'. It appears that the line λ1380·6X.U. measured by Eddy and Turner is perhaps a rough measurement of the Cu $K\beta_5$ line for which Wennerlöf has given the value 1378·24X.U.

In table II are given the wave-lengths of the 's' and 't' lines and their differences for some elements.

TABLE II
λ in X.U.

Element	't'	's'	$\Delta\lambda = (\lambda t - \lambda s)$
73Ta	1671·7	1608·6	63·1
74W	1621·6	1561·0	60·6
77Ir	1490·0	1429·5	60·5
78Pt	1449·0	1386·6	62·4
79Au	1410·0	1348·1	61·9
80Hg	1371·8	1308·5	63·3
82Pb	1305·0	1242·03	63·0
83Bi	1269·3	1209·1	60·2

It can be easily seen that $\triangle\lambda$ for all the elements is constant within experimental errors. As the forbidden lines are faint and rather diffuse, greater accuracy for the values of $\triangle\lambda$ cannot be expected. As is well known, the constancy of $\triangle\lambda$ shows that the lines 's' and 't' form a regular doublet.

Acknowledgement

The author takes this opportunity to express his deep gratitude to Professor G. B. Deodhar, for his never-failing interest and continual guidance throughout this work. Sincere thanks are also due to the Council of Scientific and Industrial Research, Government of India, New Delhi, for a research grant, without which it would have been impossible to carry out this work.

References

1. Auger, P. and Dauvillier, A. *Comptes Rendus*, 176, 1297, (1923)
2. Deodhar, G.B. and Mande, C. *Jour. Sci. & Ind. Res.*, 9B, 263, (1950)
3. Woodall, E. A. *Phil. Mag.*, 32, 1, (1941)
4. Idei, S. *Sci. Rep. Tohoku Imp. Univ.*, 19, 559, (1930)
5. Eddy, C. and Turner, H. *Proc. Roy. Soc. Lond.* (A), 114, 605, (1927)
6. Eddy. C. and Turner, H. *Proc. Roy. Soc. Lond.* (A), 111, 117, (1926)
7. Friman, E. *Zeits. f. Phy.*, 39, 813, (1926)
8. Wennerlöf, I. *Ark. Mat., Astr. Fysik.* (A), 22, Nr. 8, (1930)

A CLASS OF DIRICHLET'S SERIES POSSESSING ESSENTIAL CHARACTERISTICS OF A TAYLOR'S SERIES

By

NIRMALA PANDEY, M.A.

Research Scholar, Allahabad University.

SUMMARY

In this paper I propose to extend Hadamard's Multiplication Theorem for Taylor's series to a class of Dirichlet's series possessing essential Characteristics of a Taylor's series. Observing that the finite singularities of the series

$$h(S)=\sum_{p}^{\infty}\lambda'(n)f\{\lambda(n)\}e^{-s\lambda(n)}$$

where the functions $f(\zeta)$, $\lambda(z)$, $\zeta=re^{i\theta}$, $z=\beta+\rho e^{i\phi}$ satisfy certain conditions, are identical with those of the integral $\int_0^\infty f(\zeta)e^{-s\zeta}\,d\zeta$ whose sole finite singularity is $A-iB$ provided $\lambda(\theta)$ is of the form $A\cos\theta+B\sin\theta$, I pass on to the study of the singularities of the series

$$H(S)=\sum_{p}^{\infty}\lambda'(n)\,f_1\{\lambda(n)\}f_2\{\lambda(n)\}e^{-s\lambda(n)}$$

by means of the integral $\int_0^\infty f_1(\zeta)f_2(\zeta)\,e^{-\zeta s}\,d\zeta$ whose sole finite singularity is

$$(A_1+A_2-i\,B_1+B_2),$$

provided $\lambda_1(\theta)$ and $\lambda_2(\theta)$ are given by $\lambda_1(\theta)=A_1\cos\theta+B_1\sin\theta$,

$\lambda_2(\theta)=A_2\cos\theta+B_2\sin\theta$.

The theorem established by me in this paper is as follows:—

Theorem:—If

(1) $\lambda(z)$ be a branch of an analytic function of $z(=x+iy=\beta+\rho e^{i\phi})$, $p-1<\beta<p$ in the angle $|\phi|<\pi$ and $\lambda(z)=0\ (\rho)$ uniformly in this angle as $\rho\rightarrow\infty$;

(2) $\lambda(x)$ be an L-function such that it is positive for $x\geqslant p$, and steadily tends to infinity with x;

(3) $\lambda'(z)=0\,(1)$ as $\rho\to\infty$, uniformly in the angle $|\phi|\leqslant\pi$;

(4) $f_1(\zeta), f_2(\zeta)$ are integral functions of $\zeta=re^{i\theta}$ satisfying the relations (i) $f_1(\zeta)=0\,(e^{k_1 r})$ (ii) $f_2(\zeta)=0\,(e^{k_2 r})$, throughout the angle $|\theta|\leqslant\pi$;

(5) $$\limsup_{r\to\infty}\frac{\log|f_1(re^{i\theta})|}{r}=\lambda_1(\theta)=A_1\cos\theta+B_1\sin\theta,\ (|\theta|\leqslant\pi)$$

$$\limsup_{r\to\infty}\frac{\log|f_2(re^{i\theta})|}{r}=\lambda_2(\theta)=A_2\cos\theta+B_2\sin\theta,\ (|\theta|\leqslant\pi).$$

(6) $f_1\{\lambda(z)\}, f_2\{\lambda(z)\}$ are integral functions of z and

$$|\lambda'(z)f_1\{\lambda(z)\}f_2\{\lambda(z)\}\,e^{k\lambda(z)}\,(z-\beta)|\to 0$$

as $\rho\to\infty$ uniformly on the arc $|z-\beta|=\rho$, $|\phi|\leqslant\pi$
for some positive values of k, then the series

(7) $$H(S)=\sum_{p}^{\infty}f_1\{\lambda(n)\}f_2\{\lambda(n)\}e^{-s\lambda(n)}\,\lambda'(n)$$

represents an analytic function of S in the whole plane excepting the point

$$\overline{A_1+A_2}-i\,\overline{B_1+B_2}.$$

1. There is one important class of Dirichlet's Series, namely the series all of whose coefficients are positive for which the line of convergence is also the line of absolute convergence, and contains at least one singularity of the function represented by the series. The object of this paper is to study the properties of another class of Dirichlet's Series for which the lines of convergence and absolute convergence coincide and necessarily contain at least one singularity of the function represented by the series.

2. Theorem:—If

$\lambda(z)$ be a branch of an analytic function of $z(=x+iy=\beta+\rho e^{i\phi})$,

$$p-1<\beta<p \qquad \ldots \quad (2{\cdot}1)$$

in the angle $|\phi|\leqslant\pi$ and $\lambda(z)=0(\rho)$ uniformly in the angle as $\rho\to\infty$; $\lambda(x)$ be an L-function such that it is positive for $x\geqslant p$ and steadily tends to infinity with x; . . . (2·2)

$\lambda'(z)=0(1)$ as $\rho\to\infty$, uniformly in the angle $|\phi|\leqslant\pi$; . . . (2·3)

$f_1(\zeta), f_2(\zeta)$ are integral functions of $\zeta=re^{i\theta}$ of exponential type in

the whole plane, satisfying the relations

(i) $f_1(\zeta)=0(e^{k_1 r})$, (ii) $f_2(\zeta)=0(e^{k_2 r})$ throughout the whole plane; (2·4)

(i) $\lim_{r\to\infty} \dfrac{\log|f_1(re^{i\theta})|}{r}=\lambda_1(\theta)=A_1\cos\theta+B_1\sin\theta,\ |\theta|\leqslant\pi.$ . . . (2·5)

(ii) $\lim_{r\to\infty} \dfrac{\log|f_2(re^{i\theta})|}{r}=\lambda_2(\theta)=A_2\cos\theta+B_2\sin\theta\ ,\ |\theta|\leqslant\pi.$

$f_1\{\lambda(z)\}, f_2\ \{\lambda(z)\}$ are integral functions of z and

$|\{\lambda'(z)f_1\{\lambda(z)\}f_2\{\lambda(z)\ e^{-\kappa\lambda(z)}|(z-\beta)|\to 0$ as $\rho\to\infty$ uniformly on the arc $|z-\beta|=\rho,\ |\phi|\leqslant\pi$ for positive values of k, then the series ... (2·6)

$$H(S)=\sum_{p}^{\infty}\lambda'(n)\ f_1\{\lambda(n)\}\ f_2\ \{\lambda(n)\}\ e^{-s\lambda(n)},\quad (S=\sigma+it)\ \ldots\ (2\cdot7)$$

represents an analytic function of S in the whole plane except the point

$$S=\{(A_1+A_2)-i\ (B_1+B_2)\}.$$

Corresponding to an arbitrarily small positive number ϵ, we can choose a positive integer n_0 such that the modulus of the nth term of the series (2·7) is less than

$$\{|\lambda'(n)|e^{[\lambda_1(0)+\lambda_2(0)+\epsilon-\sigma]\lambda(n)}\}\text{ for } n\geqslant n_0(\epsilon).$$

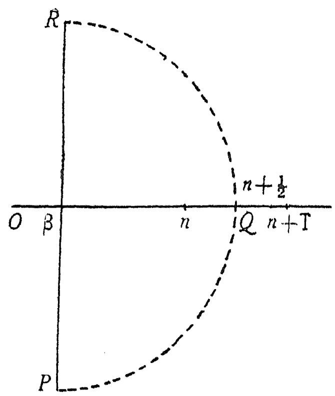
Now since $\lambda'(x)$ is itself an L-function tending steadily, by virtue of (2·3), to a definite quantity as $x\to\infty$, $\{|\lambda'(x)|e^{[\lambda_1(0)+\lambda_2(0)+\epsilon-\sigma]\lambda(x)}\}$ is a positive monotonic function which $\to 0$ as $x\to\infty$, if $\sigma\geqslant\lambda_1(0)+\lambda_2(0)+\epsilon+\epsilon'>\lambda_1(0)+\lambda_2(0)+\epsilon$. Hence the series whose nth term is $\{|\lambda'(n)|e^{(\lambda_1(0)+\lambda_2(0)+\epsilon-\sigma)\lambda(n)}\}$ will converge, if the integral

$\pm\int_0^\infty \lambda'(x)\ exp[\lambda_1(0)+\lambda_2(0)+\epsilon-\sigma]\lambda(x)\,dx$ converges. That this integral converges for $\sigma\geqslant\lambda_1(0)+\lambda_2(0)+\epsilon+\epsilon'>\lambda_1(0)+\lambda_2(0)$ is seen by putting $\lambda(x)=y$ and so the series (2·7) is absolutely and uniformly convergent in the half plane

$$\sigma\geqslant\lambda_1(0)+\lambda_2(0)+\delta>\lambda_1(0)+\lambda_2(0)$$

for every positive δ, and represents there an analytic function of S. Putting $F(z)=\lambda'(z)f_1\{\lambda(z)\}f_2\{\lambda(z)\}\ e^{-s\lambda(z)}$, and applying Cauchy's countour integration theorem, we have

$$\sum_{p}^{n}F(\nu)=\int_C\frac{F(z)}{e^{2\pi i z}-1}\,dz$$



where C is the contour formed by the lines βR and βP give respectively by $\phi = \pm\pi/2$ and the arc PQR by $|z-\beta| = n+\frac{1}{2}-\beta$, Q being on the x-axis. That is

$$\sum_{p}^{n} F(\nu) = \int_{\beta P} \frac{F(z)}{e^{2\pi iz}-1}\, dz + \int_{PQ} \frac{F(z)}{e^{2\pi iz}-1}\, dz + \int_{\beta Q} F(z)\, dz$$

$$+\int_{\beta R} \frac{F(z)dz}{e^{-2\pi iz}-1} - \int_{QR} \frac{F(z)dz}{e^{-2\pi iz}-1} \quad \ldots (2\cdot 8)$$

If S is real, positive and sufficiently large, and $\rho \to \infty$, then, by virtue of (2·6) the integrals over PQ and $QR \to 0$, since

$\left|\frac{1}{e^{-2\pi iz}-1}\right|$ and $\left|\frac{1}{e^{2\pi iz}-1}\right|$ are bounded on their respective paths of integration. Hence if S is real, positive and sufficiently large,

$$H(S) = \int_{\beta}^{\infty} F(x)\, dx + \int_{\beta}^{\infty(\pi/2)} \frac{F(z)dz}{e^{-2\pi iz}-1} + \int_{\beta}^{\infty(-\pi/2)} \frac{F(z)dz}{e^{2\pi iz}-1} \quad \ldots \quad (2\cdot 9)$$

$$= I_1 + I_2 + I_3 \text{ (say)}.$$

Considering the integral I_2, we see that

$$\left|\frac{e^{-2\pi\rho}}{e^{-2\pi i\beta}-e^{-2\pi\rho}}\right| < K_1 e^{-2\pi\rho}. \text{ on the path of integration.}$$

Also $$|F(\beta+\rho\, e^{i\pi/2})| < K_2\, e^{(k_1+k_2+|s|)|\lambda(z)|}$$
$$< K_2\, e^{(k_1+k_2+|s|)c\rho}, \quad \text{by virtue of the}$$

hypotheses of (2·1), (2·3), (2·4), where ϵ is an arbitrarily small positive number and ρ is sufficiently large.

Hence I_2 converges like

$$\int_0^\infty e^{-\rho[2\pi-(k_1+k_2+|s|)\epsilon]}\,d\rho \quad \text{which is uniformly}$$

convergent for all bounded values of s, ϵ being arbitrarily small.

Similarly we can prove that I_3 represents an integral function of s.

As regards the integral I_1, it is equal to

$$\int_\beta^\infty \lambda'(x) f_1\{\lambda(x)\} f_2(\lambda(x))\, e^{-S\lambda(x)}dx = \int^\infty f_1(\zeta) f_2(\zeta)\, e^{-S\zeta}\, d\zeta + \text{an integral}$$

function of s.

So that $H(S)-\mathcal{J}(s)=G(S)$, where $G(S)$ is an integral function of S, and $\mathcal{J}(S)$ is an analytic function of s defined initially by the integral

$$\int_0^\infty f_1(z) f_2(z)\, e^{-SZ}\, dz. \qquad . \quad . \quad (2\cdot10)$$

The equation (2·10) has been obtained on the assumption that S is real, positive and sufficiently large. But as the right-hand side represents an integral-function of S, the equation persists for all values of S. That is, the finite singularities of $H(S)$ are identical with those of $\mathcal{J}(S)$.

Considering the singularities of $\mathcal{J}(S)$ we know that the integral is absolutely and uniformly convergent in the half plane $\sigma \geqslant \lambda_1(0)+\lambda_2(0)+\delta > \lambda_1(0)+\lambda_2(0)$ where δ is any arbitrarily small positive number. Hence $\mathcal{J}(S)$ is a analytic function of S at least in the half plane $\sigma \geqslant \lambda_1(0)+\lambda_2(0)+\delta > \lambda_1(0)+\lambda_2(0)$.

Now suppose for a moment, that $S=(\sigma+it)$ lies in the region $\sigma > k+\delta$ and $t < -(k+\delta)$, where k is the maximum value of $\lambda(\theta)$ in the range $0 \leqslant \theta \leqslant \pi$. Then

$$\int_0^\infty f_1(z) f_2(z)\, e^{-SZ}\, dz = \int_0^{\infty(\theta)} f_1(z) f_2(z)\, e^{-SZ}\, dz,$$

Since $|\rho e^{i\theta} f_1(\rho e^{i\theta}) f_2(\rho e^{i\theta}) e^{-\rho e^{i\theta}(\sigma+it)}| \to 0$, as $\rho \to \infty$, is if s lies in the region given by $\sigma \cos\theta - t \sin\theta > k$, $0 \leqslant \theta \leqslant \pi$.

Now since $\displaystyle\int_0^{\infty(\theta)} |f_1(z) f_2(z)\, e^{-SZ}|\, dz \leqslant \int_0^\infty e^{[\lambda_1(\theta)+\lambda_2(\theta)+\epsilon-\sigma\cos\theta+t\sin\theta]\rho}\, d\rho$

which is uniformly convergent in the region

$$\sigma\cos\theta - t\sin\theta \geqslant \lambda_1(\theta)+\lambda_2(\theta)+\epsilon' > \lambda_1(\theta)+\lambda_2(\theta)\ , \ (\epsilon' > \epsilon, 0 \leqslant \theta \leqslant \pi)$$

the integral on the right hand side is absolutely and uniformly convergent in the region and so represents there an analytic function of S. The same result holds when θ lies between $-\pi < \theta < 0$.

By the principle of analytic continuation it follows, therefore, that the function $J(S)$ defined initially in a certain half-plane is analytic in a much wider region, namely, the region lying exterior to the curve Σ which is the envelope of the lines $\sigma\cos\theta - t\sin\theta = \lambda_1(\theta)+\lambda_2(\theta)$, $|\theta| \leqslant \pi$, i.e., the lines

$$\sigma\cos\theta - t\sin\theta = (A_1+A_2)\cos\theta + (B_1+B_2)\sin\theta, \qquad |\theta| \leqslant \pi;$$

Now this envelope is

$$(\sigma - \overline{A_1+A_2})^2 + (t+B_1+B_2)^2 = \epsilon^2$$

so that it follows that $J(S)$ is an analytic in the whole plane excepting the point $S = \{(A_1+A_2) - i(B_1+B_2)\}$.

Hence $H(S)$ is analytic everywhere with the exception of the point $S = \{(A_1+A_2) - i(B_1+B_2)\}$.

3. The function $\lambda(Z)$ contemplated in the theorem may be any one of a class of functions such as $\dfrac{Z}{\log Z}$, $\dfrac{Z}{\log\log Z}$, Z^a, $(0 < a < 1)$ for the hypothesis (2·1), (2·2), (2·3)and (2·6) are all true for such a $\lambda(Z)$. The simplest member of the class of functions satisfying the asymptotic equality (2·5) is e^z.

REMARKS

Theorem I enables us to study the singularities of a class of Dirichlet's Series in terms of those of the Laplace-Abel integral and to extand Hadamard's multiplication theorem to this class of Dirichlet's Series:—

$$h_1(S) = \sum_{p}^{\infty} f_1\{\lambda(n)\}e^{-s\lambda(n)}\lambda'(n) \text{ has singularity at } \alpha_1 = (A_1 - iB_1) \text{ and}$$

$$h_2(S) = \sum_{p}^{\infty} f_2\{\lambda(n)\}e^{-s\lambda(n)}\lambda'(n) \text{ has singularity at } \alpha_2 = (A_2 - iB_2),$$

then

$$H(S) = \sum_{p}^{\infty} f_1\{\lambda(n)\} f_2\{\lambda(n)\}e^{-s\lambda(n)}\lambda'(n) \text{ has singularity at}$$

$\{(A_1+A_2) - i(B_1+B_2)\}$.

I am indebted to Dr. P. L. Srivastava under whose guidance the result is obtained.

References

1. Hardy: The general theory of Dirichlet's Series, Theorem 10.
2. Dr. P. L. Srivastava, "A class of Dirichlet's series possessing essential characteristics of a Taylor's series." Bull. Acad. of Sciences U.P. Allahabad Vol. 3. No. 1. pp. 11-16.

ON INFINITE INTEGRALS INVOLVING PRODUCTS OF STRUVE'S FUNCTIONS

By

SNEHLATA

DEPARTMENT OF MATHEMATICS, ALLAHABAD UNIVERSITY

Communicated by Dr. P. L. Srivastava, Allahabad University

(Received on March 31, 1951)

1. The object of this paper is to evaluate some infinite integrals involving the products of Struve's functions defined by the formula

$$H_\nu(z) = \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{1}{2}z\right)^{\nu+2r+1}}{\Gamma(r+\frac{3}{2})\, \Gamma(\nu+r+\frac{3}{2})}. \qquad \ldots \qquad (1 \cdot 1)$$

It is proposed throughout the paper that the constants a and b are positive, and the parameters ν, m, n, p are, for simplicity, supposed real, the formulae are, however, valid in suitable chosen complex domains of both constants and parameters.

2. Let us suppose that

$$I_1 = \int_0^\infty x^{p-1} \,.\, e^{-ax^2} \,.\, H_{\nu_1}(bx)\, dx, \quad \nu_1 + p + 1 > 0$$

and

$$I_2 = \int_0^\infty x^{p-1} \,.\, e^{-ax^2} \,.\, H_{\nu_2}(bx)\, dx, \quad \nu_2 + p + 1 > 0.$$

Then we wish to evaluate the integral

$$I = \int_0^\infty x^{p-1} \,.\, e^{-ax^2} \,.\, H_{\nu_1}(bx) \,.\, H_{\nu_2}(bx)\, dx.$$

Substituting the value of $H_\nu(bx)$ from $(1 \cdot 1)$ we get

$$I=\int_0^\infty x^{p-1}\,.\,e^{-ax^2}\,.\sum_{r=0}^{\infty}\frac{(-1)^r(bx)^{\nu_1+2r+1}}{2^{\nu_1+2r+1}\,\Gamma(r+\frac{3}{2})\,\Gamma(\nu_1+r+\frac{3}{2})}$$

$$.\sum_{s=0}^{\infty}\frac{(-1)^s(bx)^{\nu_2+2s+1}}{2^{\nu_2+2s+1}\,\Gamma(s+\frac{3}{2})\,\Gamma(\nu_2+s+\frac{3}{2})}\,dx$$

$$=\int_0^\infty x^{p-1}\,.\,e^{-ax^2}\,.\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^r(-1)^s\,.\,(b/2)^{\nu_1+1+\nu_2+1}$$

$$\times\frac{(b/2)^{2r+2s}\,.\,x^{\nu_1+\nu_2+2+2r+2s}}{\Gamma(r+\frac{3}{2})\Gamma(s+\frac{3}{2})\Gamma(\nu_1+r+\frac{3}{2})\Gamma(\nu_2+s+\frac{3}{2})}\,dx$$

$$=(b/2)^{\nu_1+\nu_2+2}\,.\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^r\,(-1)^s\,\frac{(b/2)^{2r+2s}}{\Gamma(r+\frac{3}{2})\,\Gamma(s+\frac{3}{2})\,\Gamma(\nu_1+r+\frac{3}{2})}$$

$$\times\frac{1}{\Gamma(\nu_2+s+\frac{3}{2})}\,.\int_0^\infty x^{\nu_1+\nu_2+2r+2s+p+1}\,.\,e^{-ax^2}\,dx,$$

the inversion of summation and integration being easily justifiable.

Put $\qquad ax^2=t$ i.e. $x=\dfrac{t^{1/2}}{a^{1/2}}$

$\therefore \qquad 2ax\,dx=dt$

or $\qquad dx=\dfrac{dt}{2a^{1/2}\,.\,t^{1/2}}$

$$\therefore\; I=(b/2)^{\nu_1+\nu_2+2}\,.\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(-1)^r(-1)^s(b/2)^{2r+2s}}{\Gamma(r+\frac{3}{2})\,\Gamma(s+\frac{3}{2})\,\Gamma(\nu_1+r+\frac{3}{2})}$$

$$\times\frac{1}{\Gamma(\nu_2+s+\frac{3}{2})}\,.\int_0^\infty\left(\frac{t^{1/2}}{a^{1/2}}\right)^{\nu_1+\nu_2+2r+2s+p+1}\,.\,e^{-t}\,.\,\frac{dt}{2a^{1/2}\,t^{1/2}}$$

$$=(b/2)^{\nu_1+\nu_2+2}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(-1)^r(-1)^s(b/2)^{2r+2s}}{\Gamma(r+\frac{3}{2})\,\Gamma(s+\frac{3}{2})\,\Gamma(\nu_1+r+\frac{3}{2})\,\Gamma(\nu_2+s+\frac{3}{2})}$$

$$\times\frac{1}{2a^{\nu_1/2+\nu_2/2+p/2+r+s+1}}\,.\int_0^\infty t^{\nu_1/2+\nu_2/2+p/2+r+s+1-1}\,e^{-t}\,dt$$

$$= (b/2)^{\nu_1+\nu_2+2} \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{(-1)^r(-1)^s(b/2)^{2r+2s}}{\Gamma(r+\frac{3}{2})\,\Gamma(s+\frac{3}{2})\,\Gamma(\nu_1+r+\frac{3}{2})\,\Gamma(\nu_2+s+\frac{3}{2})}$$

$$\times \frac{\Gamma\,(\nu_1/2+\nu_2/2+p/2+r+s+1)}{2\,.\,a^{\nu_1/2+\nu_2/2+p/2+r+s+1}}$$

$$= \frac{b^{\nu_1+\nu_2+2}}{2^{\nu_1+\nu_2+3}\,.\,a^{\nu_1/2+\nu_2/2+p/2+1}} \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^r\,(-1)^s\,.\,(b/2)^{2r+2s}$$

$$\times \frac{\Gamma\,(\nu_1/2+\nu_2/2+p/2+r+s+1)}{a^{r+s}\,.\,\Gamma(r+\frac{3}{2})\,\Gamma(s+\frac{3}{2})\,\Gamma(\nu_1+r+\frac{3}{2})\,\Gamma(\nu_2+s+\frac{3}{2})}.$$

Hence, we have, for

$$a>0,\ \nu_1+\nu_2+p+2>0$$

$$\int_0^\infty x^{p-1}\,.\,e^{-ax^2}\,.\,H_{\nu_1}(bx)\,.\,H_{\nu_2}(bx)\,dx$$

$$= \frac{b^{\nu_1+\nu_2+2}}{2^{\nu_1+\nu_2+3}\,.\,a^{\nu_1/2+\nu_2/2+p/2+1}} \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^r\,(-1)^s\,(b/2)^{2r+2s}$$

$$\times \frac{\Gamma\,(\nu_1/2+\nu_2/2+p/2+r+s+1)}{a^{r+s}\,\Gamma(r+\frac{3}{2})\,\Gamma(s+\frac{3}{2})\,\Gamma(\nu_1+r+\frac{3}{2})\,\Gamma(\nu_2+s+\frac{3}{2})}.$$

Particular Cases

(i) $p=1$

$$\int_0^\infty e^{-ax^2}\,.\,H_{\nu_1}(bx)\,H_{\nu_2}(bx)\,dx = \frac{b^{\nu_1+\nu_2+2}}{2^{\nu_1+\nu_2+3}\,.\,a^{\nu_1/2+\nu_2/2+3/2}}$$

$$\times \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{(-1)^r\,(-1)^s\,(b/2)^{2r+2s}\,.\,\Gamma(\nu_1/2+\nu_2/2+\frac{3}{2}+r+s)}{a^{r+s}\,\Gamma(r+\frac{3}{2})\,\Gamma(s+\frac{3}{2})\,\Gamma(\nu_1+r+\frac{3}{2})\,\Gamma(\nu_2+s+\frac{3}{2})},$$

where $R\left(\frac{\nu_1}{2}+\frac{\nu_2}{2}+\frac{3}{2}\right)>0.$

(a) $\nu_1=\nu_2=1$

$$\int_0^\infty e^{-ax^2}\,.\,H_1(bx)\,H_1(bx)\,dx = \frac{b^4}{2^5\,.\,a^{5/2}}$$

$$\times \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(-1)^r (-1)^s (b/2)^{2r+2s} \Gamma(r+s+\frac{5}{2})}{a^{r+s} \Gamma(r+\frac{3}{2}) \Gamma(s+\frac{3}{2}) \Gamma(r+\frac{3}{2}) \Gamma(s+\frac{5}{2})}.$$

(b) $\nu_1 = \nu_2 = \frac{1}{2}$

$$\int_0^\infty e^{-ax^2} . H_{1/2}(bx) H_{1/2}(bx)\, dx = \frac{b^3}{2^4 . a^2}$$

$$\times \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(-1)^r (-1)^s (b/2)^{2r+2s} \Gamma(r+s+2)}{a^{r+s} \Gamma(r+\frac{3}{2}) \Gamma(s+\frac{3}{2}) \Gamma(r+2) \Gamma(s+2)}.$$

(c) $\nu_1 = -\nu_2$

$$\int_0^\infty e^{-ax^2} . H_{\nu_1}(bx) H_{-\nu_1}(bx)\, dx = \frac{b^2}{2^3 . a^{3/2}}$$

$$\times \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(-1)^r (-1)^s (b/2)^{2r+2s} . \Gamma(r+s+\frac{3}{2})}{a^{r+s} \Gamma(r+\frac{3}{2}) \Gamma(s+\frac{3}{2}) \Gamma(\nu_1+r+\frac{3}{2}) \Gamma(-\nu_1+s+\frac{3}{2})}.$$

Put $\nu_1 = \frac{1}{2}$ in this

$$\int_0^\infty e^{-ax^2} . H_{1/2}(bx) . H_{-1/2}(bx)\, dx = \frac{b^2}{2^3 . a^{3/2}}$$

$$\times \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(-1)^r (-1)^s (b/2)^{2r+2s} \Gamma(r+s+\frac{3}{2})}{a^{r+s} \Gamma(r+\frac{3}{2}) \Gamma(s+\frac{3}{2}) \Gamma(r+2) \Gamma(s+1)}.$$

(ii) $p = \nu_1 + 1$

$$\int_0^\infty x^{\nu_1} . e^{-ax^2} . H_{\nu_1}(bx) H_{\nu_2}(bx)\, dx = \frac{b^{\nu_1+\nu_2+2}}{2^{\nu_1+\nu_2+3} . a^{\nu_1+\nu_2/2+3/2}}$$

$$\times \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(-1)^r (-1)^s (b/2)^{2r+2s} \Gamma(\nu_1+\nu_2/2+r+s+\frac{3}{2})}{a^{r+s} \Gamma(r+\frac{3}{2}) \Gamma(s+\frac{3}{2}) \Gamma(\nu_1+r+\frac{3}{2}) \Gamma(\nu_2+s+\frac{3}{2})},$$

where
$$R\left(\nu_1 + \frac{\nu_2}{2} + \frac{3}{2}\right) > 0.$$

(a) $\nu_2 = 0$

$$\int_0^\infty x^{\nu_1} . e^{-ax^2} H_{\nu_1}(bx) H_0(bx)\, dx = \frac{b^{\nu_1+2}}{2^{\nu_1+3} . a^{\nu_1+3/2}}$$

$$\times \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(-1)^r (-1)^s (b/2)^{2r+2s} . \Gamma(\nu_1+r+s+\frac{3}{2})}{a^{r+s} \Gamma(r+\frac{3}{2}) \Gamma(s+\frac{3}{2}) \Gamma(\nu_1+r+\frac{3}{2}) \Gamma(s+\frac{3}{2})},$$

where $R(\nu_1+r+s+\frac{3}{2})>0.$

(b) $\nu_1=0$ and $\nu_2=1$

$$\int_0^\infty e^{-ax^2} . H_0(bx) H_1(bx) dx = \frac{b^3}{2^4 . a^2}$$

$$\times \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(-1)^r (-)^s (b/2)^{2r+2s} \Gamma(r+s+2)}{a^{r+s} \Gamma(r+\frac{3}{2}) \Gamma(s+\frac{3}{2}) \Gamma(r+\frac{3}{2}) \Gamma(s+\frac{5}{2})}.$$

(iii) $p=1-\nu_1$

$$\int_0^\infty x^{-\nu_1} . e^{-ax^2} . H_{\nu_1}(bx) H_{\nu_2}(bx) dx = \frac{b^{\nu_1+\nu_2+2}}{2^{\nu_1+\nu_2+3} . a^{\nu_2/2+3/2}}$$

$$\times \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(-1)^r (-1)^s (b/2)^{2r+2s} \Gamma(\nu_2/2+r+s+\frac{3}{2})}{a^{r+s} \Gamma(r+\frac{3}{2}) \Gamma(s+\frac{3}{2}) \Gamma(\nu_1+r+\frac{3}{2}) \Gamma(\nu_2+s+\frac{3}{2})},$$

where $R\left(\frac{\nu_2}{2}+\frac{3}{2}\right)>0.$

(a) $\nu_1=\frac{1}{4}$ and $\nu_2=\frac{1}{2}$

$$\int_0^\infty x^{-1/4} . e^{-ax^2} . H_{1/4}(bx) H_{1/2}(bx) dx = \frac{b^{11/4}}{2^{15/4} . a^{7/4}}$$

$$\times \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(-1)^r (-1)^s (b/2)^{2r+2s} \Gamma(r+s+\frac{7}{4})}{a^{r+s} \Gamma(r+\frac{3}{2}) \Gamma(s+\frac{3}{2}) \Gamma(r+\frac{7}{4}) \Gamma(s+2)}.$$

(b) $\nu_1=\frac{1}{2}$ and $\nu_2=1$

$$\int_0^\infty x^{-1/2} . e^{-ax^2} . H_{1/2}(bx) H_1(bx) dx = \frac{b^{7/2}}{2^{9/2} . a^2}$$

$$\times \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(-1)^r (-1)^s (b/2)^{2r+2s} \Gamma(r+s+2)}{a^{r+s} \Gamma(r+\frac{3}{2}) \Gamma(s+\frac{3}{2}) \Gamma(r+2) \Gamma(s+\frac{5}{2})}.$$

3. Now consider the integral

$$I = \int_0^\infty x^{p-1} . e^{-ax^2} . H_{\nu_1}(bx) H_{\nu_2}(bx) H_{\nu_3}(bx) dx.$$

Substituting the value of $H_\nu(bx)$ from (1·1) we get

$$I=\int_0^\infty x^{p-1}.e^{-ax^2}.\sum_{r=0}^{\infty}\frac{(-1)^r(bx)^{\nu_1+2r+1}}{2^{\nu_1+2r+1}\Gamma(r+\frac{3}{2})\Gamma(\nu_1+r+\frac{3}{2})}$$

$$.\sum_{s=0}^{\infty}\frac{(-1)^s(bx)^{\nu_2+2s+1}}{2^{\nu_2+2s+1}\Gamma(s+\frac{3}{2})\Gamma(\nu_2+s+\frac{3}{2})}$$

$$.\sum_{q=0}^{\infty}\frac{(-1)^q(bx)^{\nu_3+2q+1}}{2^{\nu_3+2q+1}\Gamma(q+\frac{3}{2})\Gamma(\nu_3+q+\frac{3}{2})}dx$$

$$=\sum_{r=0}^{\infty}\frac{(-1)^r b^{\nu_1+2r+1}}{2^{\nu_1+2r+1}\Gamma(r+\frac{3}{2})\Gamma(\nu_1+r+\frac{3}{2})}.\sum_{s=0}^{\infty}\frac{(-1)^s b^{\nu_2+2s+1}}{2^{\nu_2+2s+1}\Gamma(s+\frac{3}{2})\Gamma(\nu_2+s+\frac{3}{2})}$$

$$\times\sum_{q=0}^{\infty}\frac{(-1)^q b^{\nu_3+2q+1}}{2^{\nu_3+2q+1}\Gamma(q+\frac{3}{2})\Gamma(\nu_3+q+\frac{3}{2})}.\int_0^\infty x^{\nu_1+\nu_2+\nu_3+2r+2s+2q+p+2}e^{-ax^2}dx$$

the inversion of integration and summation being easily justifiable.

By the same process as before we have for

$$a>0,\ \nu_1+\nu_2+\nu_3+p+3>0$$

$$\int_0^\infty x^{p-1}.e^{-ax^2}.H_{\nu_1}(bx)\,H_{\nu_2}(bx)\,H_{\nu_3}(bx)\,dx$$

$$=\frac{b^{\nu_1+\nu_2+\nu_3+3}}{2^{\nu_1+\nu_2+\nu_3+4}.a^{\nu_1/2+\nu_2/2+\nu_3/2+p/2+3/2}}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{q=0}^{\infty}(-1)^r(-1)^s(-1)^q$$

$$\times\frac{(b/2)^{2r+2s+2q}\Gamma(\nu_1/2+\nu_2/2+\nu_3/2+p/2+r+s+q+\frac{3}{2})}{a^{r+s+q}\Gamma(r+\frac{3}{2})\Gamma(s+\frac{3}{2})\Gamma(q+\frac{3}{2})\Gamma(\nu_1+r+\frac{3}{2})\Gamma(\nu_2+s+\frac{3}{2})\Gamma(\nu_3+q+\frac{3}{2})}.$$

4. Generalising the result we get

$$\int_0^\infty x^{p-1}.e^{-ax^2}H_{\nu_1}(bx)\,H_{\nu_2}(bx)\ldots H_{\nu_{n-1}}(bx).H_{\nu_n}(bx)\,dx$$

$$=\frac{b^{\nu_1+\nu_2+\ldots+\nu_{n-1}+\nu_n+n}}{2^{\nu_1+\nu_2+\ldots+\nu_{n-1}+\nu_n+n+1}.a^{\nu_1/2+\nu_2/2+\ldots+\nu_{n-1}/2+\nu_n/2+p/2+n/2}}$$

$$\times \sum_{r_1=0}^{\infty} \sum_{r_2=0}^{\infty} \cdots\cdots \sum_{r_{n-1}=0}^{\infty} \sum_{r_n=0}^{\infty} (-1)^{r_1} (-1)^{r_2} \ldots (-1)^{r_{n-2}} . (-1)^{r_{n-1}} (-1)^{r_n}$$

$$\times \frac{(b/2)^{2r_1+2r_2+\ldots+2r_{n-2}+2r_{n-1}+2r_n}}{a^{r_1+r_2+\ldots+r_{n-1}+r_n} \Gamma(r_1+\frac{3}{2}) \Gamma(r_2+\frac{3}{2}) \ldots \Gamma(r_{n-1}+\frac{3}{2}) \Gamma(r_n+\frac{3}{2})}$$

$$\times \frac{\Gamma(\nu_1/2+\nu_2/2+\ldots+\nu_{n-1}/2+\nu_n/2+p/2+r_1+r_2+\ldots+r_{n-1}+r_n+n/2)}{\Gamma(\nu_1+r_1+\frac{3}{2}) \Gamma(\nu_2+r_2+\frac{3}{2}) \ldots \Gamma(\nu_{n-1}+r_{n-1}+\frac{3}{2}) \Gamma(\nu_n+r_n+\frac{3}{2})}$$

provided that

$$a>0, \; \nu_1+\nu_2+\ldots+\nu_{n-1}+\nu_n+p+n>0.$$

Acknowledgments

Thanks are due to Dr. P. L. Srivastava, D. Phil (Oxon), F. N. I., etc. of the Allahabad University for his kind guidance in preparing this paper. Thanks are also due to Dr. B. Mohan, Ph. D. of B. H. University and to the referee for their suggestions.

References

Mohan, B. Infinite Integrals involving Struve's Functions. The Quarterly Journal of Mathematics Oxford Series. Vol. 13, No. 49.

G. N. Watson, Theory of Bessel Functions (Cambridge), 1922, §10·4 (2).

NITROGEN LOSS IN SOILS ON THE APPLICATION OF AMMONIUM SULPHATE AND RETARDATION ON THE ADDITION OF COALS

By

N. R. Dhar and C. P. Agarwal

(Chemical Laboratories, University of Allahabad)

It has been concluded from a large amount of experimental observations that apart from the improvement in the physical condition of the soil, the main function of organic matter is the fixation of atmospheric nitrogen and the preservation of the soil or added nitrogen. Sugars, molasses, leaves, straw, sawdust, paper, cotton wool, lignin, oils and fats, peat, lignite and coals have been utilised as energy materials for nitrogen fixation and protection of the soil or the added nitrogen.

In this paper experimental results are recorded regarding the protective action of different coals on ammonium sulphate when mixed with soil.

Experimental Procedure. 250 gms of well dried and powdered (sieved through 1 mm sieve) garden soil was taken in shallow enamel dishes and ammonium sulphate was added with and without various types of coals to raise the carbon and nitrogen contents of the mixtures to about 1% and 0·2% respectively. Samples were analysed for its initial nitrogen and carbon percent. Two sets of each mixture were made, one was kept exposed to sunlight whilst the other was kept covered with black cloth, by the side of the exposed one. The soil mixture was daily stirred, the mixture being maintained at nearly 20% moisture by adding distilled water. The exposure was continued and the samples were taken out of the mixtures at regular intervals of 2 months. The samples were analysed for total carbon and total nitrogen according to standard methods of analysis. Mean temparature was recorded during the exposure. The dishes were daily exposed to sunlight from morning till evening (about 8-9 hours daily). The results of the analysis are recorded in the following pages:

Analysis of carbonaceous materials used

	Total Carbon % on dry basis in gms.	Total Nitrogen % on dry basis in gms.	Ash % in gms. on dry basis
Lignite	66	0·767	3·4
Lignin	60·98	0·521	0
Bituminous coal, Jharia	79·49	1·380	10·1
Bituminous coal, Assam	74·89	0·942	9·8
Anthracite, Jammu	55·35	0·854	33·7

Analysis of soil used

Total Carbon 0·514 %
Total Nitrogen 0·0645 %

Table 1

Mean temperature during the exposure 40 C
Date of starting the experiment—16-1-51
250 gms. soil + 1·64 gms. ammonium sulphate

Period of Exposure	State	Total Carbon % on dry basis in gms.	Total Nitrogen % on dry basis in gms.	Per cent of Nitrogen lost
0 day	Exposed	0·476	0·2089	...
	Covered	0·476	0·2089	...
60 days	Exposed	0·441	0·0672	68
	Covered	0·459	0·0918	56
120 days	Exposed	0·438	0·0523	75
	Covered	0·448	0·0749	64
180 days	Exposed	0·407	0·0437	79
	Covered	0·436	0·0603	71

Table 2

Mean temperature during the exposure 40°C
Date of starting the experiment—17-1-51
250 gms. Soil+1·64 gms. ammonium sulphate
+1·9 gms. Lignite (Palana Madras)

Period of Exposure	State	Total Carbon %on dry basis in gms.	Total Nitrogen %on dry basis in gms.	Per cent of nitrogen lost
0 day	Exposed	1·018	0·2037	—
	Covered	1·018	0·2037	—
60 days	Exposed	0·938	0·1040	49
	Covered	0·9586	0·1265	38
120 days	Exposed	0·881	0·0918	55
	Covered	0·912	0·1183	42
180 days	Exposed	0·824	0·0796	61
	Covered	0·871	0·1081	47

Table 3

Mean temperature during the exposure 40°C
Date of starting the experiment—18-1-51
250 gms. soil+1·64 gms. ammonium sulphate
+2 gms. Lignin.

Period of Exposure	State	Total Carbon %on dry basis in gms.	Total Nitrogen %on dry basis in gms.	Per cent of nitrogen lost
0 day	Exposed	1·030	0·2119	—
	Covered	1·030	0·2119	—
60 days	Exposed	0·955	0·0975	54
	Covered	0·973	0·1187	44
120 days	Exposed	0·899	0·0890	58
	Covered	0·913	0·1081	49
180 days	Exposed	0·824	0·0784	63
	Covered	0·853	0·0996	53

Table 4

Mean temperature during the exposure 40°C
Date of starting the experiment—19-1-51
250 gms. soil+1·6 gms. ammonium sulphate
+1·6 gms. Bituminous coal (Jharia)

Period of Exposure	State	Total Carbon %on dry basis in gms.	Total Nitrogen %on dry basis in gms.	Per cent of nitrogen lost
0 day	Exposed	1·080	0·2089	—
	Covered	1·080	0·2089	—
60 days	Exposed	1·028	0·0899	57
	Covered	1·049	0·1129	46
120 days	Exposed	0·983	0·0773	63
	Covered	1·013	0·1003	51·5
180 days	Exposed	0·948	0·0690	67
	Covered	0·975	0·0899	57

Table 5

Mean temperature during the exposure 40°C
Date of starting the experiment—20-1-51
250 gms. soil+1·64 gms. ammonium sulphate+1·6 gms.
Bituminous coal (Assam)

Period of Exposure	State	Total Carbon %on dry basis in gms.	Total Nitrogen %on dry basis in gms.	Per cent of nitrogen lost
0 day	Exposed	1·064	0·2001	—
	Covered	1·064	0·2001	—
60 days	Exposed	1·023	0·0840	58
	Covered	1·035	0·1080	46
120 days	Exposed	0·987	0·0741	63
	Covered	1·010	0·0962	52
180 days	Exposed	0·945	0·0644	68
	Covered	0·979	0·0831	58·5

Table 6

Mean temperature during the exposure 40°C
Date of starting the experiment—21-1-51
250 gms. soil+1·64 gms. ammonium sulphate
+2·3 gms. Anthracite

Period of Exposure	Sate	Total Carbon % on dry basis in gms.	Total Nitrogen % on dry basis in gms.	Per cent of nitrogen lost
0 day	Exposed	1·079	0·2017	—
	Covered	1·079	0·2017	—
60 days	Exposed	1·035	0·0784	61
	Covered	1·059	0·1005	50
120 days	Exposed	1·002	0·0668	66·5
	Covered	1·040	0·0864	57
180 days	Exposed	0·992	0·0603	70
	Covered	1·018	0·0764	62

The foregoing results clearly indicate that majority of nitrogen added to the soil in the form of ammonium sulphate (artificial fertilizer) is not retained by the soil. The loss of nitrogen from the soils in dishes receiving sunlight is always greater than in those kept covered.

The loss may be explained from the viewpoint that in the process of nitrification taking place in the soil, ammonium nitrite is produced as an intermediate substance which decomposes into nitrogen gas and water and thus causes the loss of nitrogen. The formation of ammonium nitrite from ammonium salts or proteins requires oxygen and that is why this type of denitrification should be facilitated by soil aeration.

The greater loss in the exposed sets can be explained on the basis that the oxidation and decomposition of ammonium salts is always greater in light than in the dark, and hence the possibility of formation and decomposition of ammonium nitrite is greater in light than in the dark. The results support the conclusion that sunlight plays an important part in the process of nitrification and nitrogen loss.

Dhar has in a number of papers, emphasised that the process of nitrification and nitrogen loss is chiefly a photo-chemical and surface oxidation reaction. Corbet, de Rossi and Sarkaria are in general agreement with Dhar's observations.

When carbonaceous matter like coal is added along with nitrogenous compounds to the soil, the loss of nitrogen is lowered as is clear from the tables 2-6. The results in these tables show that on addition of various types of coal with ammonium sulphate added to make the soil nitrogen content to 0·2 %, the loss of nitrogen is decreased from 68 % to 49-61 % (varying with type of coal) in 60 days, from 75 to 55-66·5 % in 120 days and from 79 % to 61-70 % in 180 days, all in light exposed set.

In a recent publication Crowther and Yates (Empire Journal of Experimental Agriculture, 1941, **9**, 77) have reported that the response of crops to phosphate and potash are substantially reduced when dung is applied but crops are equally responsive to inorganic nitrogen on dunged or undunged land. Hence organic matter enhances the value of artificial nitrogen.

The following data recently obtained at Rothamsted show clearly that both straw and cowdung when added to artificial manures produce beneficial results:—

Yield of Potatoes in tons per acre

Fertilizer	NoN, NoK	N, NoK
No farmyard manure	4·0	4·4
With ,, ,,	9·3	11·1
Response to farmyard manure	5·3	6·7

It is evident from the above table that the yield of potatoes should have been 9·7 tons (9·3 tons due to farmyard manure and 0·4 tons due to artificial nitrogen), but the actual yield is 11·1 tons. These results show that the addition of farmyard manure to artificial nitrogen produces an enhanced effect of the added mineral nitrogen.

Moreover, the following experiments on potatoes were carried on at Rothomsted for a number of years in which a given dressing of straw and nitrogen ploughed into the soil are compared with the same quantity of straw rotted with the same quantity of nitrogen in a compost heap and then applied to the land. The ploughed-in straw has always given a better yield than the compost as shown below:—

12 years (1934—45)

	Years of applying straw		Year after application	
	Compost	Straw ploughed in	Compost	Straw ploughed-in
Potatoes, tons per acre	7·86	9·40	7·40	7·97
Sugar beet sugar crop per acre	37·0	41·2	36·4	38·2
Barley, grain cwt. per acre	27·7	31·2	26·5	27·9

The above experimental results are in strong support of the conclusion drawn in 1935 by Dhar from various experiments that a mixture of ammonium sulphate and organic matter is a better fertilizer than ammonium sulphate alone.

This checking of the loss of nitrogen by the addition of carbonaceous substances can be explained from the point of view of negative catalysis, as the retarding influence of an easily oxidisable substance upon the oxidation of another substance. It is well known that the carbohydrates preserve body protein from undergoing oxidation and it is just likely that carbonaceous matters present in the soil may also be able to protect the soil protein and ammonium salt nitrogen from oxidation.

One interesting point is observed from the results that the loss is retarded maximum by lignite and least by anthracite. Hence it can be said that the more easily oxidisable the coal is, the more is the retarding effect.

It is, therefore, clear from the above results that the value of artificial fertilizers should be enhanced if they are mixed with carbonaceous substances. The greater value of the mixture of ammonium salts and carbonaceous substances for the soil than ammonium salts alone lies in the fact that not only the soil texture is improved by the colloids added with organic manure, but the carbonaceous matter acts as an agent in the partial preservation of nitrogenous compounds of the soil by behaving as negative catalyst.

Summary

The majority of nitrogen added to soil in the form of ammonium sulphate is lost from the soil due to the formation and decomposition of the unstable substance ammonium nitrite which decomposes into nitrogen gas and water, thus causing loss of nitrogen. Carbonaceous material in the form of coal when added to soil with ammonium sulphate retards the loss of nitrogen and acts as negative catalyst in the process of nitrification.

References.

Corbet, Biochem. J. 1934, **28**, 1575
Dhar, N. R., Nature 1934, **133**, 213
———, Address Nat. Acad. Sci., U. P., 1937
———, Proc. Nat. Acad. Sci., U. P., 1934, **4**, 175.
———, and G. Gopalarao, J. Indian Chem. Soc. I Ray No pp 81-91 (1933) II. of A. 1139 (1931) B 690 (1930).
———and———, Soil sci. **38**, 143-159 (19,34).
De Rossi, G., Ann. Tec. Agrar 1935
Sarkaria and Fazaluddin, Ind. J. Agric. Sci, 1933, **3**, 1057.

ON CERTAIN DIRICHLET'S SERIES

By

NIRMALA PANDEY AND GIRJA KHANNA

RESEARCH SCHOLARS, ALLAHABAD UNIVERSITY

1. The object of this paper is to study the following Dirichlet's Series:—

$$\sum_{1}^{\infty} e^{Aif(n)-S\lambda(n)} \qquad 0<A<2\pi \qquad \ldots \quad (1\cdot 1)$$

$$\sum_{1}^{\infty} e^{Aif(n)}\phi(n)\, e^{-S\lambda(n)} \qquad 0<A<2\pi \qquad \ldots \quad (1\cdot 2)$$

where the functions $f(z)$, $\phi(z)$, and $\lambda(z)$ satisfy the following conditions:—

$$e^{Aif(z)}=0(e^{-k\,|z|}) \text{ for } 0\leqslant\psi\leqslant\frac{\pi}{2},\ k_1>0 \qquad \ldots \quad (1\cdot 3)$$

$$e^{Aif(z)}=0(e^{k_2|z \sin \Psi|}) \text{ for } \frac{-\pi}{2}\leqslant\psi<0,\ 2\pi>k_2>0 \ \ldots \ (1\cdot 4)$$

$$\lambda(z)=0(|z|) \text{ as } |z|\to\infty. \qquad \ldots \quad (1\cdot 5)$$

$$\phi(z)=0\,(e^{k|z|^{\alpha}}), 0<\alpha<1,\ k>0. \qquad \ldots \quad (1\cdot 6)$$

2. Let us begin with the series $(1\cdot 1)$. It is convergent if and only if $\sigma>0$, and then absolutely convergent. In the half plane $\sigma\geqslant\delta>0$ (δ being an arbitrarily small positive number), the series represents an analytic function of S, say $H(S)$.

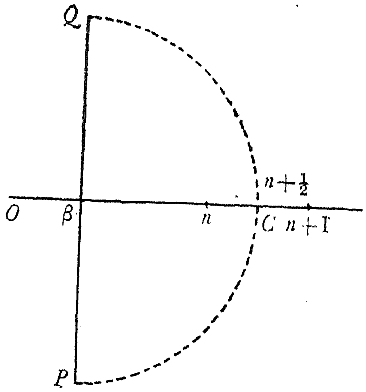
Suppose, for a moment, that S is real and positive. With β, as centre and radius $|z-\beta|=n+\frac{1}{2}-\beta$, describe the semi-circle, PCQ. Then by Cauchy's residue theorem, we have

$$\sum^{n} F(\nu) = \int_{C_1} \frac{F(z)}{e^{2\pi i z}-1}\, dz \qquad \ldots \quad (2\cdot 1)$$

where C_1 is the contour $\beta PCQ\beta$ and
$F(z) = e^{Aif(z)-S\lambda(z)}$.

Now $\left|\frac{1}{e^{2\pi i z}-1}\right| = 0(e^{-2\pi\rho\,|\sin\Psi|})$ over the arc PC.

and $\left|\frac{1}{e^{2\pi i z}-1}\right| = 0(1)$ over the arc CQ.



Therefore the modulus of the integrand over the arc PC, putting $z = \rho e^{i\Psi}$

$$= 0\left[\rho e^{-\rho(2\pi|\sin\Psi|-k_2|\sin\Psi|-\epsilon|s|)}\right] \qquad (\epsilon>0,\ k_2<2\pi)$$

which $\to 0$ as $\rho\to\infty$, S being bounded.
Similarly, the integral over the arc $CQ \to 0$, as $\rho\to\infty$, if S is bounded.

Hence for all bounded values of S, the integral over the arc $PCQ \to 0$ as $\rho\to\infty$, and we have the equation

$$H(S) = \sum^{\infty} F(\nu) = \int_{\beta}^{\infty(-\pi/2)} \frac{F(z)}{e^{2\pi i z}-1}\, dz - \int_{\beta}^{\infty(\pi/2)} \frac{F(z)}{e^{2\pi i z}-1}\, dz \ldots (2\cdot 2)$$

$$= i\int_{\beta}^{\infty} F(x)\,dx + i\int_{\beta}^{\infty} \frac{F(it)}{e^{2\pi t}-1}\, dt - i\int_{\beta}^{\infty} \frac{F(-it)}{e^{2\pi t}-1}\, dt$$

$$= I_1 + I_2 + I_3,$$

the right side giving the analytic continuation of the function $H(S)$ represented by the Dirichlet's series in the half plane $\sigma>0$.

Considering the integral I_1, we see that

$$I_1 = i\int_{\beta}^{\infty} F(x)\, dx = 0(e^{-x(k_1-|s|\epsilon)}) \quad (\epsilon>0)$$

S being bounded.

So that the integral is absolutely and uniformly convergent for all bounded values of S and hence represents an integral function of S.

Similarly I_2 represents an integral function of S.
As regards the integral I_3, the integrand is

$$=0(e^{-t(2\pi-k_2-\epsilon|s|)}) \qquad \epsilon>0.$$

so that the integral represents an integral function of S.

The equation (2·2) has been obtained on the assumption that S is real and positive. But as the right side represents an integral function of S, the equation persists for all values of S. $H(s)$ is therefore an integral function of S.

3. Let us consider the series (1·2). It is convergent if and only if $\sigma>0$ and then absolutely convergent. In the half plane $\sigma\geqslant\delta>0$, the series represents an analytic function of S say, $H(s)$.

Suppose, for a moment, that S is real and positive. Let $A>0$. Then it is easy to prove that

$$\begin{aligned} H(s) &= \sum^{\infty} e^{Aif(n)}\,\phi(n)e^{-s\lambda(n)} \\ &= i\int_0^{\infty}\frac{e^{Aif(it)}\phi(it)e^{-s\lambda(it)}}{e^{2\pi t}-1}\,dt - i\int_0^{\infty}\frac{e^{Aif(-it)}\phi(-it)e^{-s\lambda(it)}}{e^{2\pi t}-1}\,dt \\ &\qquad\qquad + i\int_0^{\infty} e^{Aif(it)}\phi(it)e^{-s\lambda(it)}\,dt \qquad \ldots \quad (3\cdot 1) \\ &= I_1+I_2+I_3. \end{aligned}$$

Since $|e^{Aif(it)}\,\phi(it)e^{-s\lambda(it)}| = 0(e^{-t(k_1-|s|\epsilon)+kt^{\alpha}}), \epsilon>0,\ \alpha<1$
both the integrals I_1 and I_3 represent integral functions of S.

As regards the integral I_2, the integrand
$=0(e^{-(2\pi-k_2-|s|\epsilon)t+kt^{\alpha}})$

so that the integral is absolutely and uniformly convergent for all bounded values of S and hence represents an integral function of S.

The equation (3·1) has been obtained on the assumption that s is real and positive, but as the right side represents an integral function of S, the equation persists for all values of S. Therefore $H(S)$ is an integral function of S.

4. According to the principle enunciated above, the following series may be defined, though expressions for their sums have not been obtained.

$$\sum_{2}^{\infty} e^{Ain - Sn/\log n} \qquad 0<A<2\pi \qquad \ldots \quad (4\cdot1)$$

$$\sum_{2}^{\infty} e^{Ain - Sn/\log\log n} \qquad 0<A<2\pi \qquad \ldots \quad (4\cdot2)$$

$$\sum_{2}^{\infty} e^{Ain + kn^{\alpha} - Sn/\log n} \qquad 0<A<2\pi,\ k>0,\ 0<\alpha<1 \quad \ldots \quad (4\cdot3)$$

$$\sum_{2}^{\infty} e^{Ain + kn^{\alpha} - Sn/\log\log n} \qquad 0<A<2\pi,\ k>0,\ 0<\alpha<1. \quad \ldots \quad (4\cdot4)$$

It is easy to prove that they represent integral functions of S.

Our thanks are due to Dr. P. L. Srivastava, under whose guidance the result is obtained.

References

1. Dr. P. L. Srivastava "On certain Dirichlet's Series". The Journal of the Indian Math. Soc. Vol XIX (1931)

2. V. F. Cowling, "Some results for Dirichlet's Series". Duke Mathematical Journal Vol 14, 1947, 907-911.

NITROGEN LOSS IN SOILS ON THE APPLICATION OF OIL CAKE AND ITS RETARDATION BY THE ADDITION OF COAL

By

N. R. DHAR AND C. P. AGARWAL

(Chemical Laboratories, University of Allahabad)

In continuation of the work published in our previous paper in which loss of nitrogen of ammonium sulphate was discussed, experiments were also carried on to study the nitrogen loss on the application of oilcake and its retardation by the addition of the same varieties of coal. In this paper some experimental observations have been recorded when oilcake was applied to soil with and without carbonaceous material in the form of various coals.

Experimental Procedure: 250 gms. of well dried and powdered (sieved through 1 m.m. sieve) garden soil was taken in shallow enamel dishes and oilcake was added with and without the various types of coals. Samples were analysed for their initial nitrogen and carbon per cent. Two sets of each mixture were made, one was kept exposed to sunlight whilst the other was kept covered with black cloth, by the side of the exposed one. The soil mixture was daily stirred, the moisture being maintained at nearly 20% by adding distilled water. The exposure was continued and the samples were taken out of the mixtures at regular intervals of two months. The samples were analysed for total carbon and total nitrogen according to standard methods of analysis. Mean temperature was recorded during the exposure. The dishes were daily exposed to sunlight from morning till evening (about 8-9 hours daily). The results of the analyses are recorded below :

Analysis of soil—Total carbon% : 0·514
Total Nitrogen% : 0·0645

Analysis of oil cake—Total carbon % : 47·3
Total nitrogen % : 6·606

Mean temperature during the exposure—40°C.

Table 1

250 gms. soil—5 gms. oil cake—starting date 25-1-51

Period of Exposure	State	Total Carbon % on dry basis in gms.	Total Nitrogen % on dry basis in gms.	Per cent of nitrogen lost
0 day	Exposed	1·343	0·188	—
	Covered	1·343	0·188	—
60 days	Exposed	1·135	0·1504	20·3
	Covered	1·087	0·1594	15·4
120 days	Exposed	0·897	0·1136	39·3
	Covered	0·934	0·1263	33
180 days	Exposed	0·665	0·0786	58·2
	Covered	0·821	0·0989	47·3

Table 2

250 gms. soil + 5 gms. oil cake + 5gms Lignite (Palana)
Starting date – 26-1-51.

Period of Exposure	State	Total carbon % on dry basis in gms.	Total Nitrogen % on dry basis in gms.	Per cent of nitrogen lost
0 day	Exposed	2·634	0·196	—
	Covered	2·634	0·196	—
60 days	Exposed	2·353	0·167	14·8
	Covered	2·308	0·175	10·7
120 days	Exposed	2·049	0·141	27·5
	Covered	2·098	0·156	20·4

180 days	Exposed	1·779	0·120	38·7
	Covered	1·937	0·141	28·5

Table 3

250 gms. soil + 5 gms. oil cake + 5 gms Lignin
Starting date—27-1-51

Period of Exposure	State	Total Carbon% on dry basis in gms.	Total Nitrogen% on dry basis in gms	Per cent of nitrogen lost
0 day	Exposed	2·552	0·189	—
	Covered	2·552	0·189	—
60 days	Exposed	2·303	0·158	16·4
	Covered	2·244	0·164	13·2
120 days	Exposed	1·948	0·130	31·2
	Covered	2·117	0·142	24·8
180 days	Exposed	1·702	0·106	43·9
	Covered	2·039	0·125	33·8

Table 4

250 gms. soil + 5 gms. oil cake + 5 gms. Bituminous coal (Nepal)
Starting date—28-1-51.

Period of Exposure	State	Total Carbon% on dry basis in gms.	Total Nitrogen% on dry basis in gms	Per cent of nitrogen lost
0 day	Exposed	1·982	0·191	—
	Covered	1·982	0·191	—
60 days	Exposed	1·763	0·157	17·8
	Covered	1·718	0·169	14·1
120 days	Exposed	1·519	0·125	34·5
	Covered	1·572	0·140	26·6

Period of Exposure	State	Total Carbon % on dry basis in gms.	Total Nitrogen % on dry basis in gms.	Per cent of nitrogen lost
180 days	Exposed	1·293	0·101	47·1
	Covered	1·468	0·122	36·1

Table 5

250 gms. soil + 5 gms. oil cake + 5 gms. Bituminous coal (Jharia)
Starting date—29-1-51

Period of Exposure	State	Total Nitrogen % on dry basis in gms	Total Nitrogen % on dry basis in gms	Per cent of nitrogen lost
0 day	Exposed	2·959	0·208	—
	Covered	2·959	0·208	—
60 days	Exposed	2·719	0·171	17·7
	Covered	2·664	0·179	13·9
120 days	Exposed	2·430	0·154	35
	Covered	2·533	0·153	26·4
180 days	Exposed	2·243	0·108	48
	Covered	2·426	0·133	36

Table 6

250 gms. soil + 5 gms. oil cake + 5 gms. anthracite
Starting date of experiment :—30-1-51

Period of Exposure	State	Total Carbon % on dry basis in gms.	Total Nitrogen % on dry basis in gms.	Per cent of nitrogen lost
0 day	Exposed	2·567	0·198	
	Covered	2·567	0·198	
60 days	Exposed	2·355	0·160	19·2
	Covered	2·313	0·168	15·1

120 days	Exposed	2·091	0·127	35·8
	Covered	2·148	0·146	27·2
180 days	Exposed	1·872	0·097	51
	Covered	2·037	0·119	39·9

The foregoing results show that considerable loss of nitrogen takes place when oil cake is added to soil. The loss of nitrogen from the soils in dishes receiving sunlight is always found greater than in those kept covered.

The data in the tables show that an addition of various types of coals with oil cake, the loss of nitrogen is decreased from 20·3 % to 14·8-19·2 % (varying with the type of coal) in 60 days, from 39·3 % to 27·5-35·8% in 120 days and from 58·2 % to 38·7-51 % in 180 days all in the light exposed set.

Comparing these results with that of ammonium sulphate (tables 2-6 of previous paper " Nitrogen loss in soils on the application of ammonium sulphate and its retardation by the addition of coal") oilcake loses much less nitrogen than ammonium sulphate chiefly because along with protein present in the oil cake, oils and fats which act as protectors of nitrogen are also present. These materials, oils and fats etc., retard the oxidation of nitrogenous compounds added to the soil. This decrease in the velocity of oxidation of nitrogenous substances lessens the possibility of formation and decomposition of the unstable intermediate compound ammonium nitrite to nitrogen gas and thus decrease the loss of nitrogen.

Summary

A marked loss of nitrogen is observed when oilcake is added to soil. This loss is accounted by the formation of the unstable substance ammonium nitrite during the oxidation of proteins. Ammonium nitrite decomposes into nitrogen gas and water, thus causing loss of nitrogen. Coals when added with oilcake partially protect the soil nitrogen as they act as negative catalyst in the process.

ON GENERALISATIONS OF LAPLACE STIELTJES TRANSFORM—I

By

SNEHLATA

DEPARTMENT OF MATHEMATICS, ALLAHABAD UNIVERSITY

Communicated by Dr. P. L. Srivastava, Allahabad Univarsity

Received on 27th September 1951

1. If

$$f(s) = \int_0^\infty e^{-st}\phi(t)\,dt,$$

where

$$\phi(s) = \int_0^\infty e^{-st}\Psi(t)\,dt,$$

then

$$f(s) = \int_0^\infty \frac{\Psi(t)}{s+t}\,dt, \qquad \ldots \quad (1\cdot1)$$

and the last equation is known as Stielties transform.

If $\Psi(t)\ dt$ be replaced by $d\alpha(t)$ a more general case of the eqn. (1·1) is obtained in the form

$$f(s) = \int_0^\infty \frac{d\alpha(t)}{s+t}. \qquad \ldots \quad (1\cdot2)$$

2. The generalisation of Stieltjes transform can be given by taking $f(s)$ to be the generalised Laplace transform of $\phi(u)$ in the form

$$f(s) = \int_0^\infty (2su)^{-1/4}\ W_{k,m}\ (2su)\phi(u)\,du$$

and $\phi(s)$ to be the generalised Laplaced transform of $\Psi(u)$ in the form

$$\phi(s)=\int_0^\infty (2su)^{-1/4} \,.\, W_{l,n}(2su)\Psi(u)du.$$

The result can be put as

THEOREM I:— If

$$f(s)=\int_0^\infty (2su)^{-1/4}\; W_{k,m}(2su)\phi(u)du$$

and

$$\phi(s)=\int_0^\infty (2su)^{-1/4}\; W_{l,n}(2su)\Psi(u)du$$

then

$$f(s)=(2s)^{-1/4}\,.\int_0^\infty (2t)^{-1/4}\Psi(t)\,.\left\{\sum_{r=0}^\infty \frac{\Gamma(-2m)(\frac{1}{2}+m-k)_r}{\Gamma(\frac{1}{2}-k-m)(2m+1)_r r!}\right.$$

$$\times\frac{(2s)^{m+r+1/2}}{(2t)^{m+r+1}}\frac{\Gamma(m+n+r+\frac{3}{2})\Gamma(m-n+r+\frac{3}{2})}{\Gamma(m-l+r+2)}$$

$$\times {}_2F_1\left[\begin{matrix} m+n+r+\frac{3}{2},\; m-n+r+\frac{3}{2} \\ m-l+r+2\end{matrix};\frac{1}{2}-\frac{s}{2t}\right]$$

$$+\sum_{r=0}^\infty \frac{\Gamma(2m)(\frac{1}{2}-k-m)_r}{\Gamma(\frac{1}{2}-k+m)(-2m+1)_r r!}\cdot\frac{(2s)^{-m+r+1/2}}{(2t)^{-m+r+1}}$$

$$\frac{\Gamma(-m+n+r+\frac{3}{2})\Gamma(-m-n+r+\frac{3}{2})}{\Gamma(-m-l+r+2)}$$

$$\left.\times {}_2F_1\left[\begin{matrix} -m+n+r+3/2,\; -m-n+r+3/2 \\ -m-l+r+2\end{matrix};\frac{1}{2}-\frac{s}{2t}\right]\right\}dt, \quad . \quad . \quad (2\cdot1)$$

provided that

$$R(\rho\pm n+5/4)>0,\; R(\pm m\pm n+3/2)>0,\; R(s)>0,\; R(\nu)>0$$

and

$$\begin{rcases} \Psi(t) = 0(t^{\rho}) \text{ for small } t \\ \quad = 0(e^{-t^{\nu}}) \text{ for large } t \end{rcases}, \quad R(\nu) > 0.$$

PROOF:—We have

$$f(s) = \int_0^\infty (2su)^{-1/4} W_{k\,m}(2su)\phi(u)du \qquad \ldots \quad (2\cdot2)$$

and

$$\phi(u) = \int_0^\infty (2tu)^{-1/4} W_{l,n}(2tu)\ \Psi(t)dt. \qquad \ldots \quad (2\cdot3)$$

Substituting the value of $\phi(u)$ from (2·3) in (2·2), we get

$$f(s) = \int_0^\infty (2su)^{-1/4} W_{k,m}(2su) \int_0^\infty (2tu)^{-1/4} W_{l,n}(2tu)\Psi(t)dt\, du.$$

On changing the order of integration, we have

$$f(s) = (2s)^{-1/4} \int_0^\infty (2t)^{-1/4}\Psi(t)dt . \int_0^\infty u^{-1/2}\ W_{k,m}(2su)\ W_{l,n}(2tu)du. \qquad \ldots \quad (2\cdot4).$$

On substituting the expression for $W_{k,m}(2su)$ in terms of Kummer's functions, we get

$$f(s) = (2s)^{-1/4} \int_0^\infty (2t)^{-1/4}\Psi(t)dt . \int_0^\infty \Big[\frac{\Gamma(-2m)}{\Gamma(\frac{1}{2}-k-m)} (2su)^{m+1/2} . e^{-su} . u^{-1/2}$$

$$\times {}_1F_1\left\{ \begin{matrix} \frac{1}{2}+m-k \\ 2m+1 \end{matrix} ; 2su \right\} + (2su)^{-m+1/2} \frac{\Gamma(2m)}{\Gamma(\frac{1}{2}-k+m)} . e^{-su} u^{-1/2}$$

$$\times {}_1F_1\left\{\begin{matrix}\frac{1}{2}-m-k; \\ -2m+1\end{matrix} 2su\right\}\Bigg] W_{l,n}(2tu)\,du.$$

Writing the equivalent series for the functions ${}_1F_1$ and integrating term by term, which is valid since
(i) the series ${}_1F_1$ is u. c. in the arbitrary interval $(0, A)$ of u,
(ii) $t^{-1/4}\, u^m\, e^{-us}\, W_{l,n}(2tu)$ and $t^{-1/4}u^{-m}e^{-us}\; W_{l,n}(2tu)$ are continuous functions of u for $u \geqslant 0$, provided that $R(m\pm n+\frac{1}{2})\geqslant 0$ and $R(-m\pm n+\frac{1}{2})\geqslant 0$, and (iii) the series obtained upon integration is u. c. in $(0, R)$, and using Goldstein's integral

$$\int_0^\infty x^{l-1}\, e^{-(\alpha^2+1/2)x}\, W_{k,m}(x)\,dx = \frac{\Gamma(l+m+\frac{1}{2})\,\Gamma(l-m+\frac{1}{2})}{\Gamma(l-k+1)}$$

$$\times {}_2F_1\left[\begin{matrix}l+m+\frac{1}{2},\; l-m+\frac{1}{2} \\ l-k+1\end{matrix};-\alpha^2\right],$$

$R(l\pm m+\frac{1}{2})>0$, $R(\alpha^2+1)>0$,
we get

$$f(s) = (2s)^{-1/4}\int_0^\infty (2t)^{-1/4}\left\{\sum_{r=0}^\infty \frac{\Gamma(-2m)}{\Gamma(\frac{1}{2}-k-m)}\cdot\frac{(\frac{1}{2}+m-k)_r}{(2m+1)_r r!}\,\frac{(2s)^{m+r+1/2}}{(2t)^{m+r+1}}\right.$$

$$\frac{\times\Gamma(m+n+r+\frac{3}{2})\,\Gamma(m-n+r+\frac{3}{2})}{\Gamma(m-l+r+2)}\;{}_2F_1\left[\begin{matrix}m+n+r+3/2,\; m-n+r+3/2 \\ m-l+s+2\end{matrix};\frac{1}{2}-\frac{s}{2t}\right]$$

$$+\sum_{r=0}^\infty \frac{\Gamma(2m)}{\Gamma(\frac{1}{2}-k+m)}\cdot\frac{(\frac{1}{2}-m-k)_r}{(-2m+1)_r r!}\cdot\frac{(2s)^{-m+r+1/2}}{(2t)^{-m+r+1}}$$

$$\frac{\Gamma(-m+n+r+\frac{3}{2})\,\Gamma(-m-n+r+\frac{3}{2})}{\Gamma(-m-l+r+2)}$$

$$\times {}_2F_1\left[\begin{matrix}-m+n+r+3/2,\; -m-n+r+\frac{3}{2} \\ -m-l+r+2\end{matrix};\; \frac{1}{2}-\frac{s}{2t}\right]\Bigg\}\Psi(t)\,dt.$$

In order to justify the change in the order of integration in (2·4) let

$$\chi(t) = \Psi(t)\,t^{-1/4}\int_0^\infty u^{-1/2}\, W_{k,m}(2su)\, W_{l,n}(2tu)\,du$$

and

$$0(u)=u^{-1/2}\, W_{k,m}\,(2su)\int_0^A t^{-1/4}\, W_{l,n}\,(2tu)\Psi(t)\,dt,$$

where A is small.

Now, $\chi(t)$ is u. c. in $t\geqslant 0$, if

$R(\pm m\pm n+\frac{3}{2})>0$ and $R(\rho\pm n+1/4)\geqslant 0$,

the behaviour of $\Psi(t)$ being given by

$$\left.\begin{aligned}\Psi(t)&=0(t^{\rho}) \text{ for small } t\\ &=0(e^{-t^{\nu}}) \text{ for large } t\end{aligned}\right\}\ , R(\nu)>0.$$

and $\theta(u)$ is u.c. in $u\geqslant 0$, provided that

$R(\rho\pm n+5/4)>0, R(\pm m\pm n+\frac{1}{2})\geqslant 0.$

Next, if we consider the integral

$$I=\int_T^\infty |u^{-1/2}\, W_{k,m}(2su)|\,du\int_{T'}^\infty |t^{-1/4}\, W_{l,n}\,(2tu)\,\Psi(t)|\,dt,$$

where T and T' are large, we find that the integral does not exceed a constant multiple of

$$\int_T^\infty |s^k u^{l+k-1/2}\,.\,e^{-su}|\,du\,.\int_{T^1}^\infty |t^{l-1/4}\,.\,e^{-ut-t^u}|\,dt$$

which tends to zero provided that $R(s)>0$, $R(\nu)>0$.

Hence the change in the order of integration is justified under the conditions

$R(\nu)>0$, $R(s)>0$, $R(\rho\pm n+1/4)\geqslant 0$, $R(\pm m\pm n+\frac{1}{2})\geqslant 0$,

which can be reduced to those stated in the theorem by the principle of analytic continuation.

Corollary:—If we put $k=l=1/4$ and $m=n=\pm 1/4$ in (2·1) we get

$$f(s) = \int_0^\infty \frac{\Psi(t)}{s+t} dt,$$

which is Stieltjes transform.

Hence Stieltjes transform is a particular case of the result (2·1).

3. Another generalisation of Stieltjes transform can be given by taking $f(s)$ to be the generalised Laplace transform of $\phi(u)$ in the form given by Meijer and Greenwood and $\phi(s)$ to be the ordinary Laplace transform of $\Psi(u)$. The result can be stated as

THEOREM II:— If

$$f(s) = \sqrt{\left(\frac{2s}{\pi}\right)} \int_0^\infty u^{1/2}. \, k_\nu(us)\phi(u)\, du$$

and

$$\phi(s) = \int_0^\infty e^{-su}\, \Psi(u)\, du,$$

then

$$f(s) = s^{-1} \int_0^\infty \Gamma(\tfrac{3}{2}-\nu)\, \Gamma(\tfrac{3}{2}+\nu)\, \frac{p^{-1}{}_{\nu-1}^{(t/s)}}{(t^2/s^2-1)^{1/2}} \Psi(t)\, dt, \quad \ldots \quad (3\cdot1)$$

provided that

$R(\beta)>0$, $R(s)>0$ and $R(\frac{3}{2}\pm\nu)>0$, $R(\rho+1)>0$,

and

$$\left.\begin{aligned} \Psi(t) &= 0(t^\rho) \text{ for small } t \\ &= 0(e^{-t^\beta}) \text{for large } t \end{aligned}\right\}, \quad R(\beta)>0.$$

PROOF:—We have

$$f(s) = \sqrt{\left(\frac{2s}{\pi}\right)} \int_0^\infty (u)^{1/2}\, k_\nu(us)\phi(u)\, du \quad \ldots \quad (3\cdot2)$$

and

$$\phi(u)=\int_0^\infty e^{-ut}\Psi(t)\,dt. \qquad \ldots \quad (3\cdot3)$$

Substituting the value of $\phi(u)$ from (3·3) in (3·2), we get

$$f(s)=\sqrt{\left(\frac{2s}{\pi}\right)}\int_0^\infty u^{1/2}k_\nu(su)\int_0^\infty e^{-ut}\Psi(t)\,dt\,du.$$

On changing the order of integration, we have

$$f(s)=\sqrt{\left(\frac{2s}{\pi}\right)}\int_0^\infty \Psi(t)\,dt.\int_0^\infty u^{1/2}\,e^{-ut}k_\nu(us)\,du. \qquad \ldots \quad (3\cdot4)$$

Now, from Watson :—Theory of Bessel Functions page 388,

we have
$$\int_0^\infty e^{-t\cosh\alpha}k_\nu(t)\,t^{\mu-1}dt$$

$$=\sqrt{\frac{\pi}{2}}\Gamma(\mu-\nu)\Gamma(\mu+\nu)\,\frac{p^{1/2-\mu}_{\nu-1/2}(\cosh\alpha)}{\sinh^{\mu-1/2}\alpha}, \qquad \ldots \quad (3\cdot5)$$

when

$R(\mu)>|R(\nu)|$ and $R(\cosh\alpha)>-1$.

Therefore, integrating with the help of the above result, we have

$$f(s)=\sqrt{\left(\frac{2s}{\pi}\right)}\int_0^\infty \Psi(t).\left\{\sqrt{\left(\frac{\pi}{2}\right)}\,\frac{\Gamma(\frac{3}{2}-\nu)\,\Gamma(\frac{3}{2}+\nu)}{S^{3/2}}\cdot\frac{p^{-1}{}_{\nu-1/2}(\cosh\alpha)}{\sinh\alpha}\right\}dt,$$

where

$R(\rho+1)>0$, $R(\beta)>0$, $R(s)>0$, $R(\frac{3}{2}\pm\nu)>0$ and $\cosh\alpha=\dfrac{t}{s}$.

This is same as (3·1).

The change in the order of integration in (3·4) can be justified exactly in the same way as in theorem I.

4. Yet another generalisation of Stieltjes transform can be given by taking $f(s)$ to be the generalised Laplace transform of $\phi(u)$ in the form given by Meijer and Greenwood and $\phi(s)$ to be the generalised Laplace transform of $\Psi(u)$ in the form

$$\phi(s)=\int_0^\infty (2\,su)^{-1/4}\,W_{l,n}(2su)\Psi(u)du.$$

The result can be stated as

THEOREM III ·—If

$$f(s)=\sqrt{\left(\frac{2s}{\pi}\right)}\int_0^\infty u^{1/2}\,K_\nu(su)\phi(u)du$$

and

$$\phi(s)=\int_0^\infty (2su)^{-1/4}\,W_{l,n}(2su)\Psi(u)du,$$

then

$$f(s)=\sqrt{s}\int_0^\infty \sum_{r=0}^\infty\left[\frac{\Gamma(-2n)}{\Gamma(\frac{1}{2}-n-l)}\,\frac{(\frac{1}{2}+n-l)_r}{(2n+1)_r r!}\cdot(2t)^{n+r+1/2}\right.$$

$$\times\frac{\Gamma(n+r-\nu+7/4)\,\Gamma(n+r+\nu+7/4)}{s^{n+r+7/4}}\cdot\frac{p_{\nu-1/2}^{-n-r-5/4}\,(t/s)}{\left(\frac{t^2}{s^2}-1\right)^{(n+r+5/4)/2}}$$

$$+\frac{\Gamma(2n)}{\Gamma(\frac{1}{2}+n-l)}\cdot\frac{(\frac{1}{2}-n-l)_r}{(-2n+1)_r r!}\cdot(2t)^{-n+r+1/2}\frac{\Gamma(-n+r-\nu+\frac{7}{4})\Gamma(-n+r+\nu+\frac{7}{4})}{s^{-n+r+7/4}}$$

$$\times \frac{p_{\nu-1/2}^{n-r-5/4}(t/s)}{\left(\frac{t^2}{s^2}-1\right)^{(-n+r+5/4)/2}}\Bigg]\cdot(2t)^{-1/4}\,.\,\Psi(t)\,dt, \qquad \ldots. \quad (4{\cdot}1)$$

provided that

$R(\beta)>0$, $R(s)>0$, $R(l\pm n+5/4)>0$, $R(\pm\nu+n+7/4)>0$, $R(\pm\nu-n+7/4)>0$,

and

$$\left.\begin{aligned}\Psi(t) &= 0(t^{\rho}) \text{ for small } t\\ &= 0(e^{-t\beta}) \text{ for large } t\end{aligned}\right\}, \ R(\beta)>0.$$

PROOF:—We have

$$f(s)=\sqrt{\left(\frac{2s}{\pi}\right)}\int_0^\infty u^{\frac{1}{2}}K_\nu(su)\phi(u)\,du. \qquad \ldots \quad (4{\cdot}2)$$

and

$$\phi(u)=\int_0^\infty (2\,tu)^{-1/4}W_{l,n}(2\,tu)\,\Psi(t)\,dt. \qquad \ldots \quad (4{\cdot}3)$$

Substituting the value of $\phi(u)$ from (4·3) in (4·2) we get

$$f(s)=\sqrt{\left(\frac{2s}{\pi}\right)}\int_0^\infty u^{1/2}K_\nu(su)\int_0^\infty (2tu)^{-1/4}W_{l,n}(2tu)\Psi(t)\,dt\,du.$$

On changing the order of integration we have

$$f(s)=\sqrt{\left(\frac{2s}{\pi}\right)}\int_0^\infty (2t)^{-1/4}\,\Psi(t)\,dt\int_0^\infty u^{\frac{1}{4}}K_\nu(su)W_{l,n}(2tu)\,du. \qquad (4{\cdot}4)$$

Substituting the value of $W_{l,n}(2tu)$ in terms of Kummers function we have

$$f(s)=\sqrt{\left(\frac{2s}{\pi}\right)}\int_0^\infty (2t)^{-1/4}\,\Psi(t)\,dt\int_0^\infty\sum_{r=0}^\infty\frac{\Gamma(-2n)}{\Gamma(\frac{1}{2}-n-l)}\frac{(\frac{1}{2}+n-l)_r}{(2n+1)_r}e^{-ut}$$

$$\times K_\nu(us)\,(2tu)^{n+r+\frac{1}{2}}u^{\frac{1}{4}}+\frac{\Gamma(2n)}{\Gamma(\frac{1}{2}+n-l)}\frac{(\frac{1}{2}-n-l)_r}{(-2n+1)_r r!}e^{-ut}$$

$$\times K_\nu(su)\,(2tu)^{-n+r+1/2}.\,u^{1/4}\Big]du.$$

Therefore, integrating term by term with the help of (3·5) which is valid since

(1) the infinite series in the integrand is u. c. in the range $u>0$, and (ii)

$$\int_0^\infty |K_\nu(su)|\,du$$

is convergent by virtue of [Watson: Theory of Bessel functions, Page 202]

$$K_\nu(z)\sim\sqrt{\left(\frac{\pi}{2z}\right)}e^{-z},$$

when z is large and $|\arg z|<\frac{3}{2}\pi$,
we have

$$f(s)=\sqrt{\left(\frac{2s}{\pi}\right)}\int_0^\infty (2t)^{-1/4}\Psi(t)\sum_{r=0}^\infty\Big[\frac{\Gamma(-2n)}{\Gamma(\frac{1}{2}-n-l)}\frac{(\frac{1}{2}+n-l)_r}{(2n+1)_r r!}(2t)^{n+r+1/2}$$

$$\times\sqrt{\left(\frac{\pi}{2}\right)}.\frac{\Gamma(n+r-\nu+7/4)\Gamma(n+r+\nu+7/4)}{s^{n+r+7/4}}\frac{P_{\nu-1/2}^{-n-r-5/4}(\cosh\alpha)}{\sinh^{(n+r+5/4)}\alpha}$$

$$+\frac{\Gamma(2n)}{\Gamma(\frac{1}{2}+n-l)}\frac{(\frac{1}{2}-n-l)_r}{(-2n+1)_r r!}(2t)^{-n+r+1/2}\sqrt{\left(\frac{\pi}{2}\right)}\frac{\Gamma(-n+r-\nu-\frac{7}{4})\Gamma(-n+r+\nu+\frac{7}{4})}{S^{-n+r+7/4}}$$

$$\times\frac{P_{\nu-\frac{1}{2}}^{n-r-5/4}(\cosh\alpha)}{\sinh^{(-n+r+7/4)}\alpha}\Big]\,dt,$$

where $R(\beta)>0$, $R(s)>0$, $R(\rho\pm n+5/4)>0$,

$R(\pm\nu+n+7/4)>0$, $R(\pm\nu-n+7/4)>0$ and $\cosh\alpha=\frac{t}{s}$.

This is same as (4·1)

The change in the order of integration in the step (4·4) can be easily justified as in theorem I.

5. We may arrive at another generalisation of Stieltjes transform by taking $f(s)$ to be the ordinary Laplace transform of $\phi(u)$ and $\phi(s)$ to be the generalised Laplace transform of $\Psi(u)$ in the form given by

$$\phi(s)=\int_0^\infty (2\,su)^{-1/4}\,W_{l\,n}\,(2su)\Psi(u)\,du.$$

The result can be stated in the form of the following theorem:—

THEOREM: IV:—If

$$f(s)=\int_0^\infty e^{-us}\,\phi(u)\,du. \qquad \ldots \quad (5\cdot1)$$

and

$$\phi(s)=\int_0^\infty (2su)^{-1/4}\,W_{l,n}\,(2su)\Psi(u)\,du, \qquad \ldots \quad (5\cdot2)$$

then

$$f(s)=\int_0^\infty (2t)^{-1}\frac{\Gamma(5/4+n)\Gamma(5/4-n)}{\Gamma(7/4-l)}$$

$$\times\,{}_2F_1\left[\begin{matrix}5/4+n,\,5/4-n\\7/4-l\end{matrix};\tfrac{1}{2}-\frac{s}{2t}\right]\Psi(t)\,dt, \qquad \ldots \quad (5\cdot3)$$

provided that

$$R(5/4\pm n)>0,\ R(t)>0,\ R(\pm n+\rho+5/4)>0,$$

and

$$\left.\begin{aligned}\Psi(t)&=0(t^{p}) \text{ for small } t\\ &=0(e^{-t^{\nu}}) \text{ for large } t\end{aligned}\right\},\ R(\nu)>0.$$

PROOF:—Proceeding in a similar manner as in theorem I the above theorem can be easily proved.

N. B. Result (5·3) has also been obtained by saksena in one of his corollaries.

References

1. Boas, Jr., R. P., "Generalised Laplace Integrals" *Bull Amer Math Soc*, **48**, 286-294 (1942).
2. Greenwood, R. E., "Hankel and other extentions of Dirichlet's series", *Annals of Math* **42** (2), 778-805, (1941)
3. Meijer, C.S., "Ueber eine Erweiterung der Laplace Transformation, I, II, ", *Proc sect sciences, K Akad.* V. Wetensc. haben te Amertendam, **43**, 599—608, 702-711, (1940).
4. Saksena, K. M., "Thesis on the Theory of Laplace stieltjes Integrals", approved for Ph. D. by the university of Agra, (1951)
5. Varma, R. S., "A generalisation of Laplace transform." current science, **16**, 17-18, (1947)
6. Varma, R.S., "On a generalisation of Laplace Integral", Na. Acad. of Sci. India. Sec. A, 1952.

ON GENERALISATIONS OF LAPLACE STIELTJES TRANS-FORM—II

By

SNEHLATA

DEPARTMENT OF MATHEMATICS ALLAHABAD UNIVERSITY

Communicated by Dr. P. L. Srivastava, Allahabad University

Received in September, 1951

I. Another* generalisation of Stieltjes transform can be given by taking $f(s)$ to be the generalised Laplace transform of $\phi(s)$ in the form given by Saksena as

$$f(s) = s^{a}\int_{0}^{\infty}(q\,su)^{c-1/2}\,.\,e^{-(p-q/2)su}\,.\,W_{k,m}(q\,su)\phi(u)du$$

and $\phi(s)$ to be the generalised Laplace transform of $\Psi(u)$ in the form

$$\phi(s) = \int_{0}^{\infty}(2su)^{-1/4}\,.\,W_{l,n}(2su)\,\Psi(u)\,du.$$

The result may be stated in the form of the following theorem:—

THEOREM V:—If

$$f(s) = s^{a}\int_{0}^{\infty}(q\,su)^{c-1/2}\,.\,e^{-(p-q/2)su}\,.\,W_{k,m}(q\,su)\phi(u)\,du$$

and

$$\phi(s) = \int_{0}^{\infty}(2su)^{-1/4}\,.\,W_{l,n}(2su)\,\Psi(u)\,du,$$

then

*This paper is in continuation to my previous paper (Snehlata, 1951)

$$f(s)=s^{a+c-1/2}q^{c-1/2}\int_0^\infty (2t)^{-1/4}\Psi(t).\left\{\sum_{r=0}^{\infty}\frac{\Gamma(-2m)}{\Gamma(\frac{1}{2}-K-m)}\cdot\frac{(\frac{1}{2}-k+m)_r}{(2m+1)_r r!}\right.$$

$$\times\frac{(qs)^{m+r+1/2}}{(2t)^{c+m+r+3/4}}\cdot\frac{\Gamma(c+m+n+r+5/4)\Gamma(c+m-n+r+5/4)}{\Gamma(c+m-l+r+7/4)}$$

$$\times{}_2F_1\left[\begin{matrix}c+m+n+r+5/4,\,c-m+n+r+5/4\\ c+m-l+r+7/4\end{matrix};\frac{1}{2}-\frac{ps}{2t}\right]$$

$$+\sum_{r=0}^{\infty}\frac{\Gamma(2m)}{\Gamma(\frac{1}{2}-k+m)}\cdot\frac{(\frac{1}{2}-k-m)_r}{(-2m+1)_r r!}\cdot\frac{(qs)^{-m+r+1/2}}{(2t)^{c-m+r+3/4}}$$

$$\times\frac{\Gamma(c-m+n+r+5/4)\Gamma(c-m-n+r+5/4)}{\Gamma(c-m-l+r+7/4)}$$

$$\left.\times{}_2F_1\left[\begin{matrix}c-m+n+r+5/4,\,c-m-n+r+5/4\\ c-m-l+r+7/4\end{matrix};\frac{1}{2}-\frac{ps}{2t}\right]\right\}dt, \quad \ldots \quad (1\cdot1)$$

provided that

$$R(c\pm m\pm n+5/4)>0, R(\rho\pm n+5/4)>0,\ R(\nu)>0,\ R(ps)>0,$$

and

$$\left.\begin{aligned}\Psi(t)&=0(t^{\rho})\ \text{for small}\ t\\ &=0(e^{-t^{\nu}}),\text{for large}\ t\end{aligned}\right\}, \qquad R(\nu)>0.$$

PROOF:—We have

$$f(s)=s^a\int_0^\infty (qsu)^{c-1/2}.e^{-(p-q/2)su}.W_{k,m}(q\,su)\phi(u)\,du \quad \ldots \quad (1\cdot2)$$

and

$$\phi(u)=\int_0^\infty (2tu)^{-1/4}.W_{l,n}(2tu)\,\psi(t)\,dt \quad \ldots \quad (1\cdot3)$$

Substituting the value of $\phi(u)$ from (1·3) in (1·2) we get

$$f(s)=s^a\int_0^\infty (qsu)^{c-1/2}.e^{-(p-q/2)su}.W_{k,m}(q\,su)\int_0^\infty (2tu)^{-1/4}.W_{k,m}(2tu)$$
$$\times\psi(t)\,dt\,du.$$

On changing the order of integration, we get

$$f(s)=s^{a+c-1/2}q^{c-1/2}\int_0^\infty (2t)^{-1/4}\Psi(t)dt.\int_0^\infty u^{c-3/4}.e^{-(p-q/2)su}.W_{k\,m}(q\,su)$$

$$\times W_{l,n}(2tu)\,du. \qquad \ldots \quad (1{\cdot}4)$$

On substituting the expression for $W_{k,m}(qsu)$ in terms of Kummer's functions we get

$$f(s)=s^{a+c-1/2}q^{c-1/2}.\int_0^\infty (2t)^{-1/4}\Psi(t)\,dt.\int_0^\infty u^{c-3/4}\,e^{-(p-q/2)su}$$

$$\times\left[\frac{\Gamma(-2m)}{\Gamma(\frac{1}{2}-k-m)}e^{-1/2.qsu}(q\,su)^{m+1/2}.{}_1F_1\left\{\begin{matrix}\frac{1}{2}+m-k\\ 2\,m+1\end{matrix};\ qsu\right\}\right.$$

$$\left.+\frac{\Gamma(2m)}{\Gamma(\frac{1}{2}-k+m)}e^{-1/2.qsu}.(qsu)^{-m+1/2}{}_1F_1\left\{\begin{matrix}\frac{1}{2}-m-k\\ -2m+1\end{matrix};qsu\right\}\right]W_{l,n}(2tu)\,du,$$

$$=s^{a+c-1/2}\,q^{c-1/2}\int_0^\infty (2t)^{-1/4}\Psi(t)\,dt\int_0^\infty e^{-psu}\left[(qs)^{m+1/2}\frac{\Gamma(-2m)}{\Gamma(\frac{1}{2}-k-m)}\right.$$

$$\times u^{c+m-1/4}\,{}_1F_1\left\{\begin{matrix}\frac{1}{2}+m-k\\ 2m+1\end{matrix};qsu\right\}+(qs)^{-m+1/2}\frac{\Gamma(2m)}{\Gamma(\frac{1}{2}-k+m)}.u^{c-m-1/4}$$

$$\left.\times{}_1F_1\left\{\begin{matrix}\frac{1}{2}-m-k\\ -2m+1\end{matrix};qsu\right\}\right]W_{l,n}(2tu)du.$$

Now, writing the equivalent series for the functions ${}_1F_1$ and integrating term by term, which is valid since

(*i*) the series ${}_1F_1$ is U.C. in the arbitrary interval $(0, A)$ of U,

(*ii*) $U^{c+m-1/4}\,e^{-psu}$ and $U^{c-m-1/4}\,e^{-psu}$

are continuous functions of u for $u\geqslant 0$, provided that $R(c+m\pm n+1/4)\geqslant 0$, $R(c-m\pm n+1/4)\geqslant 0$,

and (*iii*) the series obtained upon integration is U.C. in (O, R) we get

$$f(s)=s^{a+c-1/2}.\,q^{c-1/2}\int_0^\infty (2t)^{-1/4}\psi(t)\,.\left\{\sum_{r=0}^{\infty}\frac{\Gamma(-2m)}{\Gamma(\frac{1}{2}-k-m)}\cdot\frac{(\frac{1}{2}-k+m)_r}{(2m+1)_r r!}\right.$$

$$\times \frac{(qs)^{m+r+1/2}}{(2t)^{c+m+r+3/4}} \cdot \frac{\Gamma(c+m+n+r+5/4)\Gamma(c+m-n+r+5/4)}{\Gamma(c+m-l+r+7/4)}$$

$$\times {}_2F_1\left[\begin{matrix} c+m+n+r+5/4,\ c+m-n+r+5/4 \\ c+m-l+r+7/4 \end{matrix};\tfrac{1}{2}-\frac{ps}{2t}\right]$$

$$+\sum_{r=0}^{\infty}\frac{\Gamma(2m)}{\Gamma(\frac{1}{2}-k+m)} \cdot \frac{(\frac{1}{2}-k-m)_r}{(-2m+1)_r r!} \cdot \frac{(qs)^{-m+r+1/2}}{(2t)^{c-m+r+3/4}}$$

$$\times \frac{\Gamma(c-m+n+r+5/4)\Gamma(c-m-n+r+5/4)}{\Gamma(c-m-l+r+7/4)}$$

$$\times {}_2F_1\left[\begin{matrix} c-m+n+r+5/4,\ c-m-n+r+5/4 \\ c-m-l+r+7/4 \end{matrix};\tfrac{1}{2}-\frac{ps}{2t}\right]\Bigg\}dt. \quad . \quad . \quad . \qquad (1\cdot5)$$

In order to justify the change in the order of integration in $(1\cdot4)$ let

$$\chi(t)=\Psi(t)\,.\,t^{-1/4}\,.\int_0^\infty u^{c-3/4}\,.\,e^{-(p-q/2)su}\,.\,W_{k,m}(qsu)\,.\,W_{l,n}(2tu)\,du$$

and

$$\theta(u)=e^{-(p-q/2)su}\,.\,u^{c-3/4}\,.\,W_{k,m}(qsu)\,.\int_0^\infty t^{-1/4}\,.\,W_{l,n}(2tu)\psi(t)\,dt$$

Now $\chi(t)$ is u.c. in $t\geqslant 0$, if

$R(c\pm m\pm n+5/4)>0$ and $R(\rho\pm n+1/4)>0$,

And $\theta(u)$ is also u.c. in $u\geqslant 0$ provided that $R(\rho\pm n+5/4)>0$, $R(c\pm m\pm n+1/4)\geqslant 0$.

Next if we consider the integral

$$I=\int_T^\infty |u^{c-3/4}e^{-(p-q/2)su}\,W_{k,m}(qsu)|du\int_{T'}^\infty |t^{-1/4}W_{l,n}(2tu)\Psi(t)|\,dt$$

where T and T' are large, we find that the integral does not exceed a constant multiple of

$$\int_T^{\infty} |s^k q^{\kappa} u^{l+c+k-3/4} e^{-psu}| du \,.\, \int_{T'}^{\infty} |t^{l-1/4} \,.\, e^{-ut-t^{\nu}}|\, dt$$

which tends to zero provided that $R(ps)>0$, $R(\nu)>0$.

Hence the change in the order of integration is justified under the conditions
$R(ps)>0$, $R(\nu)>0$, $R(\rho \pm n+1/4)\geqslant 0$, $R(c\pm m\pm n+1/4)\geqslant 0$, which can be relaxed to those stated in the theorem by the principle of analytic continuation.

If we expand $W_{l,n}(2tu)$ instead of $W_{k,m}(qsu)$ in infinite series in terms of Kummer's functions in (1·4) and proceed on similar lines, we get

$$f(s) = s^{a+c-1/2} \,.\, q^{c-1/2} \,.\, \int_0^{\infty} (2t)^{-1/4} \,.\, \Psi(t)\, dt \,.\, \int_0^{\infty} u^{c-3/4} \,.\, e^{-(p-q/2)su-ut}$$

$$\times \left[\frac{\Gamma(-2n)}{\Gamma(\frac{1}{2}-l-n)} (2ut)^{n+1/2} \,.\, \sum_{r=0}^{\infty} \frac{(\frac{1}{2}+n-l)_r}{(2n+1)_r} \,.\, \frac{(2ut)^r}{r!} + \frac{\Gamma(2n)}{\Gamma(\frac{1}{2}-l+n)} \,.\, (2ut)^{-n+1/2} \right.$$

$$\left. \times \sum_{r=0}^{\infty} \frac{(\frac{1}{2}-n-l)_r}{(-2n+1)_r} \,.\, \frac{(2ut)^r}{r!} \right] W_{k\, m}(qsu)\, du$$

$$= \int_0^{\infty} \sum_{r=0}^{\infty} \left[\frac{\Gamma(-2n)}{\Gamma(\frac{1}{2}-l-n)} \,.\, \frac{(\frac{1}{2}+n-l)_r}{(n+1)_r\, r!} \,.\, \frac{(2t)^{n+r+1/4}}{q^{n+r+5/4} \,.\, s^{n-a+r+5/4}} \right.$$

$$\times \int_0^{\infty} v^{c+n+r-1/4} \,.\, e^{-\left(\frac{ps+t}{qs}-\frac{1}{2}\right)v} \quad W_{k,m}(v)\, du$$

$$+ \frac{\Gamma(2n)}{\Gamma(\frac{1}{2}-l+n)} \,.\, \frac{(\frac{1}{2}-n-l)_r}{(-2n+1)_r\, r!} \,.\, \frac{(2t)^{-n+\gamma+1/4}}{q^{-n+r+5/4} \,.\, s^{-n-a+r+5/4}}$$

$$\times \int_0^\infty v^{c-n+r-1/4} \cdot e^{-\left(\frac{ps+t}{qs}-\frac{1}{2}\right)n.} \quad W_{k,m}(v)\,dv\Big]\psi(t)\,dt$$

$$= \int_0^\infty \sum_{r=0}^\infty \Big\{ \frac{\Gamma(-2n)}{\Gamma(\frac{1}{2}-l+n)} \cdot \frac{(\frac{1}{2}+n-l)_r}{(2n+1)_r r!} \cdot \frac{\Gamma(c+m+n+r+5/4)\Gamma(c-m+n+r+5/4)}{\Gamma(c+n-k+r+7/4)}$$

$$\times \frac{(2t)^{n+r+1/4}}{q^{n+r+5/4} \cdot s^{n-a+r+5/4}} \cdot {}_2F_1 \left[\begin{matrix} c+m+n+r+5/4,\ c-m+n+r+5/4 \\ c+n-k+r+7/4 \end{matrix} ; 1-\frac{ps+t}{qs} \right]$$

$$+ \frac{\Gamma(2n)}{\Gamma(\frac{1}{2}-l+n)} \cdot \frac{(\frac{1}{2}-n-l)_r}{(-2n+1)r!} \cdot \frac{\Gamma(c+m-n+r+5/4)\,\Gamma(c-m-n+r+5/4)}{\Gamma(c-n-k+r+7/4)}$$

$$\times \frac{(2t)^{-n+r+1/4}}{q^{-n+r+5/4} \cdot s^{-n-a+r+5/4}} \cdot {}_2F_1 \left[\begin{matrix} c+m-n+r+5/4,\ c-m-n+r+5/4 \\ c-n-k+r+7/4 \end{matrix} ; 1-\frac{ps+t}{qs} \right] \Big\}$$

$$\times \Psi(t)\,dt \qquad \ldots \quad (1\cdot 6)$$

where $R(\rho \pm n+5/4)>0, R(c\pm m\pm n+\frac{5}{4})>0$,
$R(ps)>0$, $R(\nu)>0$

COROLLARY I:—

If we put $a=0$, $c=m$, $q=1$ and $p=1$ in $(1\cdot 5)$ we get

$$f(s) = s^{m-1/2} \int_0^\infty (2t)^{-1/4} \Big\{ \sum_{r=0} \frac{\Gamma(-2m)}{\Gamma(\frac{1}{2}-k-m)} \cdot \frac{\Gamma(\frac{1}{2}+m-k)_r}{(2m+1)_r r!} \cdot \frac{s^{m+r+1/2}}{(2t)^{2m+r+3/4}}$$

$$\times \frac{\Gamma(2m+n+r+5/4)\ \Gamma(2m-n+r+5/4)}{\Gamma(2m-l+r+7/4)}$$

$$\times {}_2F_1 \left[\begin{matrix} 2m+n+r+5/4,\ 2m-n+r+5/4 \\ 2m-l+r+7/4 \end{matrix} ; \frac{1}{2}-\frac{s}{2t} \right] dt$$

$$+ \sum_{r=0}^\infty \frac{\Gamma(2m)}{\Gamma(\frac{1}{2}-k+m)} \cdot \frac{(\frac{1}{2}-m-k)_r}{(-2m+1)_r r!} \cdot \frac{s^{-m+r+1/2}}{(2t)^{r+3/4}} \cdot \frac{\Gamma(n+r+5/4)\ (-n+r+5/4)}{\Gamma(-l+r+7/4)}$$

$$\times \qquad {}_2F_1\left[\begin{matrix} n+r+5/4,\ -n+r+5/4 \\ -l+r+7/4 \end{matrix} ;\tfrac{1}{2}\frac{s}{2t}\right]\Big\}\Psi(t)\,dt \qquad . \; . \; . \quad (1\cdot7)$$

which is the result given by K. M. Saksena. Vide (Thesis on the theory of Laplace Stieltjes Integrals by K. M. Saksena, Agra University, 1951)

Corollary II:—

If we put $l=1/4,\ n=\pm1/4, a=0, c=m, q=1$ and $p=1$ In (1·6) we get

$$f(s)=\frac{\Gamma(2m+1)}{s\Gamma(m-k+3/2)}\int_0^\infty {}_2F_1\left[\begin{matrix} 2m+1 \\ m-k+3/2 \end{matrix};-\frac{t}{s}\right]\Psi(t)\,dt.$$

This is the result given by Varma, R. S.

Corollary III:—

If we put $a=0,\ c=m,\ q=1, p=1$ and $k=\frac{1}{2}-m$ in (1·5) we get

$$f(s)=\int_0^\infty \frac{1}{2t}\,\frac{\Gamma(5/4+n)\,\Gamma(5/4-n)}{\Gamma(7/4-l)}\ {}_2F_1\left[\begin{matrix} 5/4+n,\ 5/4-n \\ 7/4-l \end{matrix};\tfrac{1}{2}-\frac{s}{2t}\right]\Psi(t)\,dt$$

which is a generalisation of Stieltjes transform.

If we further put $l=1/4, n=\pm1/4$ in this it becomes Stieltjes transform.

2. Yet another generalisation of Stieltjes transform can be given by taking $f(s)$ to be the generalised Laplace transform of $\phi(u)$ in the form given by Saksena,

$$f(s)=s^a\int_0^\infty (qsu)^{c-1/2}.e^{-(p-q/2)su}.W_{k,m}(qsu)\phi(u)\,du$$

nad $\phi(s)$ to be the generalised Laplace transform of $\Psi(u)$ in the form

$$\phi(s)=\int_0^\infty (us)^{n-1/2}e^{-us/2}W_{l,n}(us)\Psi(u)\,du.$$

The result can be put in the form of the following theorem :—

THEOREM VI:—If

$$f(s)=s^a\int_0^\infty (qsu)^{c-1/2}.e^{-(p-q/2)su}.\ W_{k,m}(qsu)\phi(u)\,du$$

and

$$\phi(s)=\int_0^\infty (us)^{n-1/2}.\ e^{-us/2}.\ W_{l,n}(us)\Psi(u)\,du,$$

then

$$f(s)=s^{a+c-1/2}.q^{c-1/2}\int_0^\infty t^{n-1/2}\left\{\sum_{r=0}^{\infty}\frac{\Gamma(-2m)}{\Gamma(\frac{1}{2}-k-m)}\frac{(\frac{1}{2}-k+m)_r}{(2m+1)_r\,r!}\right.$$

$$\times\frac{(qs)^{m+r+1/2}}{t^{c+m+n+r+1/2}}.\ \frac{\Gamma(c+m+r+2n+1)\,\Gamma(c+m+r+1)}{\Gamma(c+m+n-l+r+3/2)}$$

$$\times{}_2F_1\left[\begin{matrix}c+m+r+2n+1,\ c+m+r+1\\ c+m+n-l+r+3/2\end{matrix};-\frac{ps}{t}\right]+\sum_{r=0}^{\infty}\frac{\Gamma(2m)}{\Gamma(\frac{1}{2}-k+m)}$$

$$\times\frac{(\frac{1}{2}-m-k)_r}{(1-2m)_r\,r!}.\ \frac{(qs)^{-m+r+1/2}}{t^{c-m+n+r+1/2}}.\ \frac{\Gamma(c-m+2n+r+1)\,\Gamma(c-m+r+1)}{\Gamma(c-m+n-l+r+3/2)}$$

$$\left.\times{}_2F_1\left[\begin{matrix}c-m+2n+r+1,\ c-m+r+1\\ c-m+u-l+r+3/2\end{matrix};-\frac{ps}{t}\right]\right\}\Psi(t)\,dt\quad\ldots\quad(2\cdot1)$$

provided that

$R(c\pm m\pm 2n+1)>0,\ R(c\pm m+1)>0,\ R(\rho+n\pm n+1)>0,$
$R(ps)>0,\ R(\nu)>0,$

and the behaviour of $\Psi(t)$ being given by

$$\left.\begin{aligned}\Psi(t)&=0(t^{\rho})\text{ for small }t\\ &=0(e^{-t^{\nu}})\text{ for large }t\end{aligned}\right\},\ R(\nu)>0.$$

PROOF:—We have

$$f(s)=s^a\int_0^\infty (qsu)^{c-1/2}\,.\,e^{-(p-q/2)su}\,.\,W_{k,m}(qsu)\phi(u)\,du \quad . \; . \; . \quad (2\cdot2)$$

and

$$\phi(u)=\int_0^\infty (ut)^{n-1/2}\,.\,e^{-ut/2}\,W_{l,n}(ut)\Psi(t)\,dt. \quad . \; . \; . \quad (2\cdot3)$$

On substituting the value of $\phi(u)$ from (2·3) in (2·2) we get

$$f(s)=s^a\int_0^\infty (qsu)^{c-1/2}\,.\,e^{-(p-q/2)su}\,W_{k,m}(qsu)\int_0^\infty (ut)^{n-1/2}e^{-ut/2}$$

$$\times W_{l,n}(ut)\Psi(t)\,dt\,du.$$

On changing the order of integration, we get

$$f(s)=s^{a+c-1/2}\,.\,q^{c-1/2}\int_0^\infty t^{n-1/2}\,\Psi(t)\,dt\,.\int_0^\infty u^{c+n-1}\,e^{-\{ps-(qs-t)/2\}u}$$

$$\times W_{k,m}(qsu)\,W_{l,n}(u)\,du \quad . \; . \; . \quad (2\cdot4)$$

Now, substituting the expression for $W_{k,m}(qsu)$ in terms of Kummers' function and proceeding in a similar manner as in Theorem V this theorem VI can be easily proved.

The change in the order of integration in the step (2·4) can be justified exactly in the same way as in theorem V.

If, however, we substitute the value of $W_{l,n}(ut)$ in terms of Kummers' functions in (2 4) and proceed on similar lines, we shall get

$$(f)=s^{a+c-1/2}\,.\,q^{c-1/2}\,.\int_0^\infty t^{n-1/2}\,.\Psi(t)dt\int_0^\infty\Big\{\sum_{r=0}^{\infty}\frac{\Gamma(-2n)}{\Gamma(\frac{1}{2}-l-n)}$$

$$\times \frac{(\frac{1}{2}+n+l)_r}{(2n+1)_r} (-1)^r \frac{u^{c+2n+r-1/2}}{r!} t^{n+r+1/2} \,.\, e^{-(p-q/2)su} \,.\, W_{k,m}(qsu)$$

$$+\sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \frac{\Gamma(2n)}{\Gamma(\frac{1}{2}-l+n)} \cdot \frac{(\frac{1}{2}-n+l)_r}{(-2n+1)_r} u^{c+r-1/2} \,.\, t^{-n+r+1/2} \,.\, e^{-(p-q/2)su}$$

$$\times W_{k,m}(qsu)\}du$$

$$= s^{a+c-1/2} \,.\, q^{c-1/2} \,.\, \int_0^\infty t^{n-1/2}\Psi(t) \,.\, \Big\{ \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \cdot \frac{\Gamma(-2n)}{\Gamma(\frac{1}{2}-l-n)}$$

$$\times \frac{(\frac{1}{2}+n+l)_r}{(2n+1)_r} \cdot \frac{t^{n+r+1/2}}{(qs)^{c+2n+r+1/2}} \cdot \frac{\Gamma(c+2n+m+r+1)\,\Gamma(c+2n-m+r+1)}{\Gamma(c+2n-k+r+3/2)}$$

$$\times {}_2F_1\left[\begin{matrix} c+2n+m+r+1,\ c+2n-m+r+1 \\ c+2n-k+r+3/2 \end{matrix} ; 1-p/q\right]$$

$$+\sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \frac{\Gamma(2n)}{\Gamma(\frac{1}{2}-l+n)} \cdot \frac{(\frac{1}{2}-n+l)_r}{(1-2n)_r} \frac{t^{-n+r+1/2}}{(qs)^{c+r+1/2}}$$

$$\times \frac{\Gamma(c+r+m+1)\Gamma(c+r-m+1)}{\Gamma(c+r-k+3/2)}$$

$$\times {}_2F_1\left[\begin{matrix} c+m+r+1,\ c-m+r+1 \\ c-k+r+3/2 \end{matrix} ; 1-p/q\right]\Big\} dt \qquad \ldots \quad (2{\cdot}5)$$

where $R(\nu)>0$, $R(ps)>0$, $R(c+2n\pm m+1)>0$, $R(c\pm m+1)>0$, $R(n+n+\rho+1)>0$.

COROLLARY:—

If we put $a=0$, $c=m$, $q=1$ and $p=1$ in (2·5) we get

$$f(s) = \frac{\Gamma(-2n)\Gamma(2n+1)\Gamma(2m+2n+1)s^{-2n-1}}{\Gamma(\frac{1}{2}-n-l)\Gamma(m+2n-k+3/2)}$$

$$\times \int_0^\infty t^{2n} {}_2F_1\left[2m+2n+1,\ n+l+\tfrac{1}{2};\ m+2n-k+\tfrac{3}{2}; -\frac{t}{s}\right]\Psi(t)\,dt$$

$$+\frac{\Gamma(2n)\Gamma(2m+1)\,s^{-1}}{\Gamma(\frac{1}{2}+n-l)\Gamma(m-k+3/2)} \cdot \int_0^\infty {}_3F_2\left[\begin{matrix} 2m+1,\ 1,\ l-n+\frac{1}{2} \\ m-k+3/2,\ -2n+1 \end{matrix} ; -t/s\right]\psi(t)\,dt. \qquad \ldots \quad (2{\cdot}6)$$

which is the result obtained by Saxena Vide (Thesis in the Theory of Laplace Stieltjes Integrals by K. M. Saxena, Agra' Varsity, 1951).

References

1. Saksena, K. M. "Thesis on the Theory of Laplace Stieltjes Intergrals," approved for Ph.D. by the University of Agra, (1951).
2. Varma, R.S., "A generalisation of Laplace transform," Current Science, **16**, 17—18, (1947).
3. Varma, R.S., "On a generalisation of Laplace Integral, Proc. the National Academy of Sciences, India, Sec. A, Vol. 20, 1951.

ON GENERALISATIONS OF LAPLACE STIELTJES TRANSFORM-III

By

SNEHLATA

DEPARTMENT OF MATHEMATICS, ALLAHABAD UNIVERSITY.

Communicated by Dr. P. L. Srivastava

Received on 27th September 1951.

1. In this paper it is proposed to give two theorems which connect the generalised Hankel transform of $f(s)$ with $\phi(t)$ where $f(s)$ is the generalised Laplace transform of $\phi(t)$ in the form

$$f(s)=s^{a}\int_{0}^{\infty}(qst)^{c-1/2}\,e^{-(p-q/2)st}.W_{l,n}\,(qst)\phi(t)\,dt.$$

THEOREM 1 :— If

$$f(s)=s^{a}\int_{0}^{\infty}(qst)^{c-1/2}.\,e^{(p-q/2)st}.\,W_{l,n}(qst)\phi(t)\,dt \qquad \ldots \qquad (1\cdot1)$$

and

$$g(x)=(\tfrac{1}{2})^{\lambda}\int_{0}^{\infty}(xy)^{\lambda+1/2}J_{\lambda}^{\mu}\left(\frac{x^2y^2}{4}\right)f(y)\,dy \qquad \ldots \qquad (1\cdot2)$$

where

$J_{\lambda}^{\mu}(x)$ is Maitland's function defined by

$$J_{\lambda}^{\mu}(x)=\sum_{r=0}^{\infty}\frac{(-x)^{r}}{r!\,\Gamma(1+\lambda+\mu r)}, \qquad \mu>0,$$

then

$$g(x)=(\tfrac{1}{2})^{\lambda}\, q^{-(\lambda+a+3/2)} \int_0^{\infty} \sum_{r=0}^{\infty} \Big[t^{-(\lambda+a+2r+3/2)} \frac{(-1)^r x^{\lambda+2r+1/2}}{2^{2r} r!\, \Gamma(1+\lambda+\mu r)}$$

$$\times \frac{q^{-2r}\Gamma(\lambda+c+n+a+2r+3/2)\, \Gamma(\lambda+c-n+a+2r+3/2)}{\Gamma(\lambda+c-l+a+2r+2)}$$

$$\times {}_2F_1 \left\{ \begin{matrix} \lambda+c+n+a+2r+\frac{3}{2},\ \lambda+c-n+a+2r+\frac{3}{2} \\ \lambda+c-l+a+2r+2 \end{matrix} ;\, 1-p/q \right\} \Big] \phi(t)\, dt, \qquad \dots \quad (1{\cdot}3)$$

provided that

$R(\rho+c\pm n+1)>0$, $R(\lambda+c+a\pm n+3/2)>0$, $R(\beta)>0$, $R(p)>0, \mu>0$,

and
$$\left.\begin{aligned} \phi(x) &= 0(x^{\rho}) \text{ for small } x \\ &= 0(e^{-x^{\beta}}) \text{ for large } x \end{aligned}\right\},\ R(\beta)>0.$$

PROOF :—We have from (1·1) and (1·2)

$$g(x) = (\tfrac{1}{2})^{\lambda} \int_0^{\infty} (xy)^{\lambda+1/2} . \mathcal{J}_{\lambda}^{\mu}\left(\frac{x^2y^2}{4}\right) . y^a . \int_0^{\infty} (qyt)^{c-1/2} . e^{-(p-q/2)yt}$$

$$\times\, W_{l,n}(qyt)\phi(t)dt\, dy$$

$$= (\tfrac{1}{2})^{\lambda} . q^{c-1/2} . \int_0^{\infty} t^{c-1/2}\phi(t)\, dt . \int_0^{\infty} (xy)^{\lambda+1/2} \mathcal{J}_{\lambda}^{\mu}\left(\frac{x^2y^2}{4}\right)$$

$$\times\, y^{c+a-1/2} e^{-(p-q/2)yt} .\ W_{l,n}(qyt)dy, \qquad \dots \quad (1{\cdot}4)$$

on changing the order of integration.

To justify the change in the order of integration let,

$$\theta(y) = (xy)^{\lambda+1/2} \mathcal{J}_{\lambda}^{\mu}\left(\frac{x^2y^2}{4}\right) y^{c+a-1/2} \int_0^{A} t^{c-1/2} . e^{-(p-q/2)yt}$$

$$\times\, W_{l,n}\,(qyt)\phi(t)dt$$

and

$$\chi(t) = t^{c-1/2}\phi(t) . \int_0^\infty (xy)^{\lambda+1/2} . \mathcal{J}_\lambda^\mu\left(\frac{x^2y^2}{4}\right) y^{c+a-1/2}$$

$$\times e^{-(p-q/2)yt} . W_{l,n}(qyt)\, dy,$$

where A is small.

Now $\theta(y)$ converges uniformly in $y \geqslant 0$, if $R(\lambda+c\pm n+a+\frac{1}{2}) \geqslant 0$, $R(\rho+c\pm n+1)>0$,

as $\qquad \mathcal{J}_\lambda^\mu(x) = 0(1). \qquad (x \to 0)$.

And $\chi(t)$ converges uniformly in $t \geqslant 0$, if $R(c+\rho\pm n) \geqslant 0$, $R(\lambda+c+a\pm n+3/2)>0$.

Again if we consider the integral

$$I = \int_T^\infty |(xy)^{\lambda+\frac{1}{2}} \mathcal{J}_\lambda^\mu\left(\frac{x^2y^2}{4}\right) y^{c+a-\frac{1}{2}}|dy . \int_{T'}^\infty |t^{c-\frac{1}{2}} e^{-(p-q/2)yt}$$

$$\times W_{l,n}(qyt)\phi(t)|\, dt$$

where T and T' are large, we find, on account of the following asymptotic behaviour of $\mathcal{J}_\lambda^\mu(x)$:—

$$\mathcal{J}_\lambda^\mu(x) = 0\left[x^{-k'(\lambda+1/2)} . \exp\left\{(\mu x)^{k'} . (\cos \pi k')/\mu k'\right\}\right], \qquad (x \to \infty) \qquad \ldots \quad (1 \cdot 4A)$$

where $k' = \dfrac{1}{1+\mu}$,

that I does not exceed a constant multiple of

$$\int_0^\infty \left|(xy)^{\lambda+1/2} . (x^2y^2/4)^{-k'(\lambda+1/2)} . y^{c+a+l-1/2} . e^{\mu^{k'}} . (x^2y^2/4)^{.k'}(\cos \pi k')/\mu k' \right| dy$$

$$\times \int_{T'}^\infty |t^{l+c-1/2} . e^{-pyt-t^\beta}|\, dt,$$

which tends to zero when $0<\mu<1$, $R(\beta)>0$, $R(p)>0$.

Hence the change in the order of integration is justified when $R(\beta)>0$, $1>\mu>0$, $R(c+a\pm n+\rho)\geqslant 0$, $R(\lambda+c+a\pm n+\frac{1}{2})\geqslant 0$, $R(p)\geqslant 0$,
which can be relaxed to those mentioned in the theorem by the principle of analytic continuation.

Now,

$$\int_0^\infty (xy)^{\lambda+1/2} \cdot \mathcal{J}_\lambda^\mu\left(\frac{x^2y^2}{4}\right) y^{c+a-1/2} \cdot e^{-(p-q/2)yt} \cdot W_{l,n}(qyt)\, dy$$

$$=\int_0^\infty \sum_{r=0}^\infty \frac{x^{\lambda+1/2}\left(-\frac{x^2}{4}\right)^r}{r!\,\Gamma(1+\lambda+\mu r)}\left[y^{\lambda+2r+c+a} \cdot e^{-(p-q/2)yt} \cdot W_{l,n}(qyt)\right] dy,$$

on substituting the infinite series for $\mathcal{J}_\lambda^\mu\left(\frac{x^2y^2}{4}\right)$.

Integrating term by term, which is permissible. This is equal to

$$\sum_{r=0}^\infty \frac{(-1)^r x^{\lambda+2r+1/2}}{2^{2r} r!\,\Gamma(1+\lambda+\mu r)} \cdot \frac{\Gamma(\lambda+c+n+a+2r+2/3)\,\Gamma(\lambda+c-n+a+2r+2/3)}{\Gamma(\lambda+c-l+a+2r+2)}$$

$$\times (qt)^{-(\lambda+c+a+2r+1)}\; {}_2F_1\left[\begin{matrix}\lambda+c+n+a+2r+\frac{3}{2},\ \lambda+c-n+a+2r+\frac{3}{2}\\ \lambda+c-l+a+2r+2\end{matrix};\ 1-p/q\right].$$

Therefore,

$$g(x)=(\tfrac{1}{2})^\lambda \cdot q^{-(\lambda+a+3/2)}\int_0^\infty \sum_{r=0}^\infty \Bigg\{t^{-(\lambda+a+2r+3/2)}\,\frac{(-1)^r x^{\lambda+2r+1/2}\, q^{-2r}}{2^{2r} r!\,\Gamma(1+\lambda+\mu r)}$$

$$\times \frac{\Gamma(\lambda+c+n+a+2r+3/2)\,\Gamma(\lambda+c-n+a+2r+3/2)}{\Gamma(\lambda+c-l+a+2r+2)}$$

$$\times {}_2F_1\left[\begin{matrix}\lambda+c+n+a+2r+3/2,\ \lambda+c-n+a+2r+3/2\\ \lambda+c-l+a+2r+2\end{matrix};\ 1-p/q\right]\Bigg\}\phi(t)\, dt \qquad \ldots \quad (1\cdot4B)$$

provided that $R(\rho+c\pm n+1)>0$, $R(\lambda+c+a\pm n+3/2)>0$, $R(\beta)>0$, $\mu\geqslant 0$, $R(p)>0$.

CASE 1:—If we put $a=0$, $q=1$, $c=m$, $l=k$, $n=m$ and $p=1$ in (1·4*B*) we get

$$g(x)=(\tfrac{1}{2})^{\lambda}\int_{0}^{\infty}\sum_{r=0}^{\infty}\Big[t^{-(\lambda+2r+3/2)}\frac{(-1)^{r}\cdot x^{\lambda+2r+1/2}}{2^{2r}r!\,\Gamma(1+\lambda+\mu r)}$$

$$\times\frac{\Gamma(\lambda+2m+2r+3/2)\,\Gamma(\lambda+2r+3/2)}{\Gamma(\lambda+m-k+2r+2)}\Big]\phi(t)\,dt,$$

which is the result obtained by Saksena.

CASE 2:—If we put $a=0$, $q=1$, $c=m$, $l=k$, $n=m$, $p=1$, $\lambda=\nu$ and $\mu=1$ in (1·4B) we get

$$g(x)=\frac{\Gamma(2m+\nu+3/2)\,\Gamma(\nu+3/2)}{\Gamma(m-k+\nu+2)\,\Gamma(\nu+1)}(\tfrac{1}{2})^{\nu}\cdot x^{\nu+1/2}\cdot\int_{0}^{\infty}t^{-\nu-3/2}\cdot\Psi(t)$$

$$\times{}_{4}F_{3}\left[\begin{matrix}m+\nu/2+3/4,\ m+\nu/2+5/4,\ \nu/2+3/4,\ \nu/2+5/4\\ m/2-k/2+\nu/2+1,\ \ m/2-k/2+\nu/2+3/2,\ v+1\end{matrix};-\frac{x^{2}}{t^{2}}\right]dt,$$

which is again the result obtained by Saksena.

2. THEOREM 2:—If

$$f(s)=\int_{0}^{\infty}(2st)^{-1/4}\cdot W_{k,m}(2st)\phi(t)\,dt \qquad \ldots \quad (2{\cdot}1)$$

and

$$g(x)=(\tfrac{1}{2})^{\lambda}\cdot\int_{0}^{\infty}(xy)^{\lambda+1/2}\cdot\mathcal{J}_{\lambda}^{\mu}\left(\frac{x^{2}y^{2}}{4}\right)f(y)\,dy \qquad \ldots \quad (2{\cdot}2)$$

then

$$g(x)=(\tfrac{1}{2})^{\lambda}\int_{0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(-1)^{r}x^{\lambda+2r+1/2}t^{-\lambda-2r-3/2}}{2^{2r-s-1/4}r!\,s!\,\Gamma(1+\lambda+\mu r)}$$

$$\times\left[\frac{\Gamma(-2m)\,(\tfrac{1}{2}+m-k)_s\Gamma(\lambda+m+2r+s+7/4)\,2^m}{\Gamma(1/2-k-m)\,(2m+1)_s}\right.$$
$$\left.+\frac{\Gamma(2m)(1/2-m-k)_s\Gamma(\lambda-m+2r+s+7/4)\cdot 2^{-m}}{\Gamma(1/2-k+m)\,(-2m+1)_s}\right]\phi(t)\,dt \quad \ldots \quad (2\cdot 3)$$

provided that $R(\rho\pm m+5/4)>0$, $R(\lambda\pm m+7/4)>0$, $R(\beta)>0$, $\mu>0$ and

$$\left.\begin{aligned}\phi(x)&=0(x^{\rho}) \text{ for small } x\\ &=0(e^{-x^{\beta}}) \text{ for large } x\end{aligned}\right\},\ R(\beta)>0.$$

PROOF:—

Proceeding as in theorem no. 1 this theorem can be easily proved.

References

1. Agarwal, R. P., "Thesis on the Theory of Hankel transform and self reciprocal Functions", approved for Ph. D. by the university of Lucknow, (1949).
2. Saksena, K. M., "Thesis on the Theory of Laplace stieltjes Integrals" approved for Ph. D. by the university of Agra, (1951).
3. Varma, R. S., "A generalisation of Laplace transform," Current Science, **16**, 17-18, (1947).

INSTRUCTIONS TO CONTRIBUTORS

Articles should be as *brief* as possible. The viewpoint should be comprehensive in giving the relation of the paper to previous publications of the author or of others and in exhibiting, when practicable, the significance of the work for other branches of science. Elaborate technical details of the work and long tables of data should be avoided, but authors should be precise in making clear the new results and should give some record of the methods and data upon which they are based.

Manuscripts should be prepared with a current number of the PROCEEDINGS as a model in matters of form and should be typewritten in duplicate with double spacing, the author retaining one copy. Illustrations should be confined to text-figures of simple character, though more elaborate illustrations may be allowed in special instances to authors willing to pay for their preparation and insertion. Authors are requested to send in all drawings or other illustrations in a state suitable for direct photographic reproduction. They should be drawn on a large scale in Indian ink on Bristol board, with temporary lettering in pencil. Great care should be exercised in selecting only those that are essential. If unsatisfactory drawings are submitted authors may be required to have them redrawn by a professional artist. Particular attention should be given to arranging tabular matter in a simple and concise manner.

References to literature, numbered consecutively, will be placed at the end of the article and short footnotes should be avoided. It is suggested that references to periodicals be furnished in some detail and in genral in accordance with the standard adopted for the Subject Catalogue of the International Catalogue of Scientific Literature, *viz.*, name of author, with initials following (ordinarily omitting title of paper) abbreviated name of journal, volume, year, inclusive pages.

Papers by members of the Academy may be sent to the General Secretary, National Academy of Sciences, India , Allahabad ,U. P. Papers by non-members of the Academy should be submitted through some member.

No pape s exceeding twelve pages of printed matter will be published in the Proceedings except with a special permission of the Council.

Every paper must be accompanied by *three copies* of a brief summary not exceeding 300 words in length.

Proof will ordinarily be sent and the author is requested to return it with the manuscript at his earliest convenience. All proof corrections involve heavy expenses which would be negligible if the papers are carefully revised by the authors before submission.

PRINTED BY SADAN LAL, B.Sc., AT THE TECHNICAL PRESS,
2, LAJPAT ROAD, ALLAHABAD